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ABSTRACT

The Education Technology Center (ETC) Fractions Group works to investigate well-documented difficulties that children have in understanding fractions. A central assumption underlying this group's inquiry is that students' difficulties in manipulating fractions arise from their lack of understanding about the nature of fractions. In order to learn operations with fractions, students must first understand the order properties of fractions and their nature as "numbers between numbers." The notion of betweenness, so evident in the perception of continuous quantity, is neither evident nor even salient in the symbol systems and notation schemes used to describe continuous quantity. Yet, understanding betweenness appears to be a prerequisite for effective manipulation of fractions. From a larger view, betweenness is an important theme pervading much of mathematical thinking. Thus, this project's focus on teaching students to understand betweenness is chosen not only as a means of addressing student difficulties with fractions, but also as a way of teaching a fundamental concept in mathematics. This document includes: (1) discussion of the development of teaching materials; (2) lesson plans; (3) pilot study results; (4) anecdotal notes; and (5) guidelines for observers used in the pilot study. A list of 43 references is appended. (CW)

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AN INQUIRY INTO CHILDREN'S UNDERSTANDING OF THE NATURE OF FRACTIONS

Technical Report

November 1985

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Technical Report

November 1985

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OVERVIEW

Three important views concerning the understanding of fractions lie at the heart of the research problem:

> From the perspective of mathematics as a discipline, the problem of fractions may be seen as the problem of quantifying continuous quantity and of devising a symbol system and a notation scheme that encode that quantification.

> From a classroom perspective, the problem of fractions is a problem of teaching a symbol system (e.g. fraction bar), a complicated set of notational schemes (numerator and denominator), and computational algorithms within these notational schemes.

> From a cognitive developmental perspective, the problem of fractions can be seen as a problem of mapping reliably the perceived properties of the continuous quantities being described onto the symbol system and notation scheme, and vice versa.

The problem being addressed by the ETC Fractions Group is students' well-documented difficulty understanding and using fractions. A central assumption underlying this group's inquiry is that students' difficulties in manipulating fractions arise from their lack of understanding about the nature of fractions. In order to learn operations with fractions, students must first understand the order properties of fractions and their nature as "numbers between numbers." The notion of betweenness, so evident in the perception of continuous quantity, is neither evident nor even salient in the symbol systems and notation schemes used to describe continuous quantity. Yet, understanding betweenness appears to be a prerequisite for effective manipulation of fractions. From a larger view, betweenness is an important theme pervading much of mathematical thinking. Thus, this project's focus on teaching students to understand betweenness is chosen not only as a means of addressing student difficulties with fractions, but also as a way of teaching a fundamental concept in mathematics.

For example, the technique of successive decimal approximations to square roots and the view of the area of a circle being between the areas of inscribed and circumscribed polygons are instances of betweenness which are experienced early in the study of mathematics. At advanced levels, the notion of continuity is even more central to mathematical thinking, reaching perhaps the fullest flower in the differential and integral calculus. Arguments of continuity, convergence, and boundedness all depend on some form of the idea of betweenness. Furthermore, the idea of betweenness is not limited to classical areas of mathematics. For example, the fractal dimensions work of Mandelbrot is a remarkable example of the liberating power of being able to think about spatial dimensions whose dimensionality lies between the integers.

There are, of course, many ways of representing fractional parts of entities. Among the more commonly used representations are the linear model we are exploring, an area model, a pseudo-area model (pie charts), and fractional parts of discrete sets. Traditional wisdom maintains that in learning a new concept, students should be presented with a rich set of different representations. The premise is that these varied representations will allow students to see the new concept in as many different guises as possible and thus strengthen their grasp of what they are just learning. This view has frequently influenced efforts to teach fractions, yet most of these efforts are not very effective with students. The approach of the ETC Fractions Group stems from a belief that multiple representations may be more confusing than enlightening because of the lack of correspondence across representations of the order properties of fractions. The Fractions Group approach uses a single model, the linear model, and focuses on the concepts of order, betweenness, and equivalence. (In fact, it is interesting to observe that the latter concept can be viewed as the limiting case of betweenness: equivalent fractions are those between which there is no number.)

Based on a review of existing materials and prior research in this area, the Fractions Group has designed experimental lessons with several criteria in mind. First, a single model should be used in order to reduce conceptual complexity. Second, the linear model should be chosen because it most clearly represents fractions as numbers for measuring continuous quantities. Third, the curriculum should build on the use of rulers, tools that students readily recognize as familiar and useful for measuring lengths. Fourth, the curriculum should combine manipulable materials, with which students gain immediate "hands-on" experience, and computer activities that extend the experience beyond the limits achievable with physical objects.

The teaching experiment consists of four modules of lessons in the following sequence:

Stage 1 Simple Length Measurement: Integer Number of Units

The purpose of this part of the teaching procedure is to emphasize the notion that the numerical magnitude of a continuous quantity is entirely dependent on the unit that is used and may be expressed as an approximation to the actual length.

Stage 2 Length Measurement: Non-Integer (Less Than One) Number of Units

The purpose of the second module is for students to learn how to make rulers by folding paper strips, to use the rulers to measure continuous quantities less than one and to experience equivalent fractions by labeling and comparing arrays of number lines.

Stage 3 Length Measurement: Non-Integer (Greater Than One) Number of Units

The emphasis at this stage is on equivalence and ordering of both fractions and mixed numbers and on the use of fractional rulers in the measurement of continuous quantities longer than one whole unit. The ruler activities are facilitated on the microcomputer which allows extensions to cases not possible with hands-on materials. The idea of betweenness is stressed by using a sequence of rulers with successively finer subdivisions to define the length in question with greater precision.

Stage 4 Order Relationships of Fractions and the Notion of Betweenness

The purpose of this stage is for students to examine aspects of order and betweenness within the notation systems for fractions; microcomputer-based games that depend on the order properties of fractions have been developed for this purpose. These games and activities provide opportunities for students to explore a wide variety of strategies for ordering fractions by size, and to practice them.

In December 1984 and May 1985, the lessons for the modules were pilot tested with small groups of fifth and seventh grade students. These tests were undertaken to determine, on an informal basis, whether the students appeared to engage in and understand the tasks, whether the lessons appeared to be a promising curriculum for achieving the research goals of this group, and to provide information for the revision of the materials. The first two questions were clearly answered in the affirmative. The students at both grade levels responded with interest to the series of activities and appeared (on the basis of informal observations) to gain insights about fractions. The pilot test experience indicated the need for some revisions of the lessons, to omit redundant activities, clarify confusing ones, and add some activities to ensure a clear conceptual progression for students. This report describes the research literature, teaching activities, and results of the first two pilot studies.

During the Fall, 1985, members of the group will revise the modules as indicated and develop assessment instruments including both pre- and post-tests and clinical interview guides. The clinical interview process will be pilot tested with several of the students who participated in the May, 1985, pilot study. In the spring, 1986, the experimental curriculum will be taught to four small groups of students in order to clarify the students' conceptions of fractions and to assess in more detail the utility of the lessons in helping students develop a clearer understanding of the order properties of fractions.

Analysis of the Group's work during the 1985-86 Academic Year will be presented in a report planned for September, 1986.

REVIEW OF RESEARCH LITERATURE

General Approach

Many of the concerns and viewpoints of the ETC Fractions Group are supported in the research literature. The NAEP (National Assessment of Education Progress) reports show that students experience significant difficulty learning and applying fractional concepts (Carpenter et al., 1976; Carpenter et al., 1980). Peck and Jencks (1981) express the need to shift the emphasis from the learning of rules for operations on fractions to the awareness of a conceptual basis. Behr et al. state that "school programs tend to emphasize procedural skills and computational algorithms for rational numbers " (1983, p. 91) rather than providing a "careful development of important functional understandings" (1983, p.92).

Wearne-Hiebert and Hiebert also feel that "we need to spend more time in the early grades developing foundational concepts...[and that]...we are progressing too rapidly into the operations with fractions without ensuring a full understanding of the initial concepts" (1983, p. 105).

Similary Skypek strongly states that "unless students possess a...stable and robust concept of 'this new kind of number' (as opposed to whole numbers), little meaningful learning of how to operate with them can occur" (1984, p.12).

In arguing for mathematics teachers and mathematics educators to reduce students' views of mathematics as a bag of tricks to produce right answers, Post (1981) addresses some features of successful attempts at developing meanings and understandings:

- active student involvement
- the use of manipulative materials
- the opportunity for students to talk together and with their teacher
- the emphasis on conceptual meanings prior to symbols, operations, and algorithms.

Payne (1976) points out that beginning around 1973, research shifted from the comparison of various algorithms to an examination of what children learn while being taught a carefully developed sequence. His research and that of his colleagues, often referred to as the Michigan Studies, focused on measuring the quality and durability of the cognitive structure that students were building within a specific context.

Ellerbruch and Payne (1978) claimed that both research and classroom practice suggest that the introduction to fraction concepts be made through a single model and they recommend the part - whole measurement model as the most natural for young children. Their Initial Fraction Sequence (IFS) involved students in making models from folded strips of paper and proceded from concrete or pictorial models to oral fraction names, to natural language written names and finally to formal mathematical symbols. The ETC Fractions

Group has followed a similar progression and has found it useful. Its major departure from the Michigan work has been to use strips and number lines as rulers for actual measurement in the early stages of the development of fractional concepts. Used this way, the need for fractions comes out of a measurement context.

The Rational Numbers Project funded by the National Science Foundation (Behr et al., 1980; Lesh et al., 1980; Post et al., 1982; Behr et al., 1983; Behr et al., 1984; Post et al., 1985) has been concerned with the identification and description of mental processes employed by children engaged in various types of theory-based instruction. The project was designed to provide information which will help describe qualitative shifts in children's thinking about rational numbers (Lesh et al., 1980). According to Behr et al. (1983), the project incorporates three interacting components, an instructional component, an evaluation component, and a diagnostic-remedial component. It draws heavily on the work of Payne (1976) and Kieren (1976).

One of the most important pieces of literature on fractions is Kieren's (1976) analysis of the components of the concept of rational numbers. He feels that rational numbers can be interpreted in at least six ways, referred to as subconstructs: as part-to-whole comparisons; as a measure of continuous or discrete quantity; as a decimal; as a ratio; as an indicated division (quotient); and as an operator. He contends that a complete algebraic understanding of rational numbers requires both an understanding of each of these separate subconstructs and their interrelationships. His theory is sometimes referred to as Mathematical Variability (Post et al., 1982). To put the work of the ETC Fractions Group into the perspective of this model, its focus is on the interrelationships between the part - whole and measurement subconstructs in the context of continuous quantity as the most meaningful starting point for children.

In delineating an operational definition of the concept of a rational number, Post et al. not only use the Mathematical Variability Model of Kieren (1976, 1981), but they also use a Perceptual Variability Model based on work by Dienes (1963) to illustrate various physical embodiments. These are listed as follows (Post et al., 1982).

- Counters, egg cartons and other <u>set - subset</u> interpretations.

- Rods, number lines, and other <u>length</u> interpretations.

- Tiles, graph paper, geoboard and other <u>area</u> interpretations.

- Symbolic algorithms.

Regarding the use of Dienes' Perceptual Variability scheme, Post et al. (1982) feel that different materials may be useful at different points in the development of a particular rational number subconstruct. To put the work of the ETC Fractions Group into perspective in this regard, its focus is on length interpretations, such as number strips, rods, number lines, and rulers, all used as physical embodiments in the context of fractions in

part-to-whole comparisons and as measures of continuous quantities (See the enclosed chart showing the focus of the ETC Fractions Group research in relation to the Mathematical/Perceptual Variability Schemes).

The research of the ETC Fractions Group involves an interplay of hands-on experience and technological approaches. The group plans to use two commercial programs and three pieces of software under development to investigate children's understanding of the order and betweenness properties of fractions and the use of fractions as measurement quantities.

The most flexible environment for the manipulation of fractions within the measurement representation will be a microcomputer based number lines environment. The ETC Fractions Group is developing a "toolkit" of programs that allows the student or teacher to order fractions on number lines that are juxtaposed, to interpolate any number of number lines between juxtaposed number lines, to graphically add and subtract parts of number lines, and to project parts of number lines onto parallel number lines thereby permitting a graphical representation of the operations of multiplication and division of fractions. The capabilities of this sort of environment extend well beyond those necessary for the exploration of the research currently being addressed. However, it is consistent with ETC's software design policy to produce software tools that are grade-wide in their applicability and that can be used broadly.

Mathematical Variability Scheme
Kieren's (1976) Subconstructs

Mathematical/Perceptual Variability Scheme for Fractions

Dienes' (1963) Perceptual Variability Scheme
(Sample Physical Embodiments)

	Counters, egg cartons as *set-subset* interpretations	Rods, number lines, rulers as *length* interpretations	Tiles, graphs, geoboard as *area* interpretations	Symbolic Algorithms
Part-to-Whole Comparisons				
Measure of *continuous* quantity				
or *discrete* quantity				
Decimal				
Ratio				
Quotient				
Operator				

Shaded regions indicate the focus of the ETC Fractions Group research

Specific Activities

In the pages to follow, specific aspects and details of past research findings will be related to each stage of the ETC Fraction Group's activities.

Stage 1 Simple Length Measurement: Integer Number of Units

Bohan (1971) states that in the selection of a model, one must make a choice which will provide transfer from concrete stages to the abstract concepts which it was chosen to support. Whole number measurement activities were chosen as a starting point by the ETC Fractions Group because they set the stage for the later work on order and betweenness and also demonstrate the need for fractions in the measurement of continuous quantity.

These initial measurement activities show the arbitrary nature of the unit of length as well as the total dependency of the numerical magnitude of a measurement on the particular unit of length used. They foster the notion of betweenness in a concrete setting since any measurement which students obtain lies between two consecutive whole number of units of length. The need for fractional subdivisions arises naturally from the desire for more precise measurements. The fact that measurement activities lead to the abstract concept of order is strongly endorsed by Kieren who states that "... measurement is the natural locus for considering order. All of the order properties (e.g. trichotomy, anti-symmetry, transitivity) arise naturally out of measurement situations" (1976, p. 131).

The first stage of the ETC Fraction Group's measurement activities involves students in reporting the length of an object as being "a little more than eight pencils" or "between ten and eleven blackboard erasers." These measurement activities motivate the need for subdividing the unit of measurement in order to express a length more precisely by means of fractional parts.

Stage 2 Length Measurement: Non-Integer (Less Than One) Number of Units

The folding of paper strips into equal segments has been strongly endorsed as a model for the conceptualization of fractions (Bohan 1971; Payne 1976; Van de Walle and Thompson 1984; Scott 1981; Kieren 1976; Ellerbruch and Payne 1978). Bohan (1971) found that paper folding activities promoted good understanding of equivalent fractions and were especially effective with students in generating equivalent fractions in higher terms. Payne (1976) comments on the benefits that children derive from folding strips of adding machine tape to compare equivalent fractions.

Van de Walle and Thompson (1984) state that when fractional concepts are expressed in terms of fraction strips, students are motivated and challenged to attend to the meaning of the fractions involved.

Scott (1981) relates the numerous advantages of paper folding activities. Among them, he states that through the use of folded strips the relationship to the unit is immediate, that the arbitrary nature of the choice of unit is apparent, that size comparisons can be easily made for the ordering of fractions, that equivalent fractions can be readily generated, and that they aid in the conceptualization of fractional operations. Although he proposed techniques for folding fractional parts other than halves, thirds, fourths, sixths, and eighths (for example, fifths with 2 1/2 loops), the ETC Fractions Group chose to generate the fifths, sevenths, ninths, and tenths rulers through an optional projection activity rather than through paper folding techniques.

Kieren (1976) offers helpful insights to be considered when a student uses the measurement interpretation of rational numbers:

1. The student must realize that the unit should be held invariant when the subdivisions are made and that one may partition the unit into any number of congruent parts.

2. The student must be able to conceptualize part - whole relationships and recognize the occurrence of equivalence arising from the various partitionings of a consistent length.

3. The student must develop the concept of an order relationship and relate the proper symbols to the physical reality.

When relating the fractions on a folded strip to points on a number line, Ellerbruch and Payne (1978) stress the need to emphasize that you are recording the length of the strip from the beginning to the endpoint of the piece. The ETC Fractions Group also found that students needed to be shown how to write the fractions at the subdivision marks rather than internal to the intervals. These two researchers give strong endorsement to the measurement model by stating that "A measurement model seems most natural to children....In a measurement model, parts are compared to the whole unit and a number is attached to represent the part" (1978, p.130).

In the ETC activities for this stage, students fold paper strips to produce rulers with half, third, fourth, sixth, and eighth subdivisions. Students use their rulers to draw fractional lengths and to measure continuous quantities less than one whole unit. Particular attention is paid to the issue of equivalent fractions through the use of vertical comparisons in an array of number lines. The idea of betweenness is also introduced.

Stage 3 Length Measurement Units: Non-Integer (Greater Than One) Number of Units

Part-Whole Relationships

Working under the premise that children need a better foundation in the meaning of fractions rather than more work with rules and procedures for computation, Van de Walle and Thompson (1984) conducted a series of activities on part - whole and whole - part relationships. Using Cuisenaire

rods, their students did the following types of tasks:

1. Given a whole, find a unit fraction.

2. Given a unit fraction, find the whole.

3. Given a unit fraction, find a non-unit fraction with the same denominator.

4. Given a non-unit fraction, find a unit fraction with the same denominator.

5. Given a non-unit fraction, find the whole.

6. Given a whole, find any other fraction.

The ETC Fractions Group included these tasks in the May 1985 pilots and found that the students were able to deal with the issue of reversibility. Some students were even able to go one step further:

> Given a non-unit fraction, find a non-unit fraction with a different (but related) denominator.

Thus the ETC activities include the six tasks used by Van de Walle and Thompson plus this additional step.

Wearne-Hiebert and Hiebert (1983) state that they feel that children's later difficulties with fractions can be traced to an incomplete understanding of fundamental part - whole relationships. In conducting research with 6th and 8th grade students where both error analysis and sequence analysis were utilized, they found that tasks which involved finding a whole given a part were no more difficult than the more conventional task of finding a part given a whole (for fractions less than one).

In an earlier study conducted by Hiebert and Tonnessen (1978), the part - whole interpretations of Piaget, Inhelder, and Szeminska (1960) were viewed to see if they are suitable for both discrete (set/subset) and continuous (length and area) representations of fractions. Piaget et al. had devised seven subconcepts as a classification scheme to code and categorize students' responses and to describe the students' developmental levels of understanding regarding the fraction concept. These subconcepts include the realization that:

a. the whole is divisible into separate parts,
b. a fraction implies a determinate number of parts,
c. the subdivisions must be exhaustive,
d. there is a fixed relation between the number of parts and the number of subdivisions,
e. the parts have a nesting or hierarchical character,
f. the whole is conserved under subdivision,
g. all parts must be equal.

The research conducted by Hiebert and Tonnessen (1978) showed that these seven subconcepts are appropriate for continuous quantity representations, but not for discrete representations. Behr et al. (1983) feel that these continuous quantity tasks require a well-developed anticipatory scheme. In contrast, discrete tasks can be solved without treating the set as a whole and without anticipating the final solution. They conclude that cognitive structures involved in solving rational number problems referring to a discrete model would be different from those referring to a continuous model.

Since the ETC Fractions Group is working only with a continuous model, these seven subconcepts are relevant. The mixed number measurement activities in Stage 3 will serve as a good measure of subconcept (e) above regarding the nesting character of fractions. With the far too brief experience in the May pilot study with the Mixed Number Measurement Software, several students tried, for example, an eighths ruler followed by a fourths ruler without the realization that if the eighths ruler had not worked, the fourths ruler also would not work. They did not realize that fourths are embedded in the eighths. Students would need more time than one or two partial sessions to make this observation. The new Numberlines Software under development will also help students with this important subconcept. The other six subconcepts seemed to be well taken care of in the activities even as hastily taught as in the May pilot study.

Number Lines Activities

According to Kieren (1976), the number line embodies the measure subconstruct of rational numbers. This measure subconstruct suggests that the number line can conceptually be treated as a ruler. In the number line model, a reference point is used to establish a unit, and the unit is subdivided into equal parts. For example, if one wants to measure with a ruler to the nearest one-tenth, the unit would be subdivided into tenths. The measure subconstruct also includes an iteration of units in a continuous fashion for lengths beyond one. In the number line model, there is no visual separation between consecutive units, as in the typical region model where space is left between copies of the unit. Hence fractional number lines serve well as rulers to measure continuous quantities of length.

Only very preliminary research has been done on the number line model for fractions. Novillis (1976) analyzed the fraction concept into sixteen related subconcepts each associated with one of the three models--geometric regions, sets, and number lines. She found that associating a proper fraction with a point on a number line was more difficult for intermediate students than associating a proper fraction with a part - whole model where the unit was a geometric region or a set. She suggests using one or both of the two easier models as a pre-requisite to associating a fraction with a point on a number line. Payne (1976) also reported that elementary students had more difficulty with the number line model than with an area part - whole model. Muangnapoe (1975) reported third and fourth graders exhibiting the problem of not attending to the scaling. The ETC Fractions Group feels that experiences with folded strips and rods are not only necessary prerequisites for number line work but also should be related directly to the building of number lines and rulers.

In the study done by Novillis (1976), she used number lines of length one, two, and three. When the length was greater than one, some students disregarded the scaling and treated whatever number line was depicted as a whole regardless of its length. In a follow-up study, Novillis-Larson (1980) presented seventh graders with tasks involving the location of fractions on number lines that were one and two units long and for which the number of subdivisions either equaled or were twice the denominator of the fraction. She found that associating a fraction with a point was significantly easier on a number line of length one than of length two. She also found that students had difficulty associating a fraction like 3/4 with a point for which the partitioning suggests 6/8.

Using the Novillis-Larson Test (1980), Behr et al. (1983) found that children differed in how well they identified the unit on a number line. Their data also showed that problems in which the subdivisions of the unit did not equal the denominator of the fraction were more difficult for students to solve than were problems in which the subdivisions equalled the denominator. They also found that the necessary, but not directly used, subdivision markings on a number line acted as perceptual distractors for some students. The most difficult case of perceptual distractors involved inconsistent and partial cues where more than one type of subdivision was shown but only partially on the same number line.

In viewing this research, the ETC Fractions Group feels that the difficulties with the number line model reported in the literature may stem from the use of the model often in contexts that are contrived and unnatural to the use of number lines as rulers. In constrast, Chaffe-Stengel and Noddings (1982) found number lines very successful as a model. In their research, they used number lines to teach fractions as numbers that can be "counted" in a measurement sense and to demonstrate equivalent fractions through vertical projections on a set of carefully scaled lines. Their work concurs more with the findings so far of the ETC Fractions Group.

The work of the ETC Fractions Group is both an extension of and a departure from the research reported on number lines. The emphasis of the group is not on labeling an isolated point on an isolated number line, but rather on using an array of number lines to view properties of equivalence, order, and betweenness and on using number lines as rulers for the measurement of length.

Initially students label points on number lines by placing their paper folded strips along lines marked with the corresponding subdivisions of the unit. Students are encouraged to label points using the actual denominators related to the subdivisions. That is, the point 6/8 on the line sudivided in eighths would be labeled 6/8, not 3/4. The equivalence of 6/8 and 3/4 would then be shown visually by lining up the markings vertically on a sheet of carefully scaled number lines. Students use the set of pictured number lines to name equivalent fractions, to discuss which of two fractions is smaller (or larger), and to find a fraction between two given fractions.

In the Numberlines Games, the players actively move markers designated fractional lengths along the number lines. They may use the particular fraction in one of its equivalent forms or split it up into component fractional lengths to maintain a winning strategy. At the end of Stage 3,

students use number line rulers to measure lengths to various levels of precision. They record their measurements, for example, as being between 2 1/3 and 2 3/4, or being between 2 3/8 and 2 4/8, or being close to 2 4/5.

Pilot studies by the ETC Fractions Group indicate that this expanded view of the use of number lines may help students gain an understanding of equivalence, order, and betweenness. An array of fractional number lines showing denominators from halves through tenths will give students an opportunity to experience visually these relationships. The ETC Fractions Group is also planning to utilize this model to fuller potential through an interactive, exploratory computerized number lines environment.

Mixed Number Measurement

The literature is very sparse on the use of fractions in the context of mixed number measurement as defined by the ETC Fractions Group. Even though Novillis-Larson (1980) presented seventh graders with tasks involving number lines greater than one unit, the labeling was not done in the spirit of measurement and was not an exploration of the betweenness properties. The measure subconcept of Kieren (1976) offers an important challenge for research and development on these topics.

The ETC activities at this stage involve students using fractional rulers to measure continuous quantities longer than one whole unit. Ruler activities are then facilitated on the microcomputer which allows extensions to cases not possible with hands-on materials. The idea of betweenness is stressed in this piece of software through the use of a sequence of rulers with successively finer subdivisions to define the length in question with greater precision.

Stage 4 Order Relationships of Fractions and the Notion of Betweenness

Prior to the work done by Peck and Jencks (1981) and by the Rational Numbers Project funded by the National Science Foundation (Behr, et al., 1984; Post et al., 1985), very little attention had been paid to children's ability to deal with the ordering of fractions. Some groundwork had been laid on the issue of equivalence of fractions by means of tasks requiring proportional reasoning (E.F. Karplus et al., 1974; Noelting and Gagne, 1980; R. Karplus et al., 1983).

One important aspect of the research of Noelting and Gagne (1980) which will be useful to the ETC Fractions Group is the listing of a Piagetian based scheme to categorize proportional reasoning and fraction comparison tasks, now called the "Noelting Levels."

Level 1: two unequal rationals varying either the first component or the second component, but not both, e.g. 1/3 vs. 2/3 or 4/5 vs. 4/7.

Level 2A: two equal unit rationals, e.g. 1/3 vs. 2/6.

Level 2B: two equal non-unit rationals, e.g. 4/5 vs. 8/10.

Level 3A: two unequal rationals with at least one pair of "between" or "within" components divisible, e.g. 2/3 vs. 4/5 with divisibility "between" 2 and 4 or 2/5 vs. 3/9 with divisibility "within" the 3/9.

Level 3B: two unequal rationals with no "between" or "within" component divisibility, e.g. 2/3 vs. 3/5.

These levels may be helpful to the ETC Fractions Group as categorization schemes in its development of pre-test and post-test experiences dealing with equivalence, ordering, and betweenness.

In conducting a clinical teaching experiment on order and equivalence of rational numbers with 12 fourth graders, Behr et al. (1984) examined the responses of the students to find similarities and differences in their approaches. The major concern of the study was the description of the thought processes or solution strategies of the students. Three classes of fractions were used:

- same numerator, whether unit fraction or non-unit fractions
- same denominator
- different numerator and denominator.

The strategies used by the children were categorized as follows:

1. attention to the denominator for the same numerator tasks and to the numerator for the same denominator tasks,

2. the application of a ratios strategy when the numerator and denominator differ,

3. thinking that depended on manipulative aids,

4. thinking that compared the fractions in question to a third fraction or a whole number reference, and then used the transitive property,

5. thinking influenced by the knowledge of whole numbers (a debilitating influence when the numerators are the same or when both numerators and denominators differ and a facilitating influence when the denominators are the same).

They found that children whose rational number concepts were insecure tended to have the continuing influence from their knowledge of whole numbers. They concluded that:

> "... the understanding of order and equivalence of fractions requires an understanding of the compensatory relation between size and number of equal parts in a partitioned unit. The variations in children's ability to attain this understanding has implications for instruction and is an important concern for research." (pp. 337-338).

The challenge that they pose is indeed one of the major concerns of the ETC Fractions Group.

Peck and Jencks (1981) cite further confusion that some sixth grade children demonstrated. Four out of the twenty students interviewed sketched incorrect diagrams for 2/3 and 3/4 and said that the fractions were the same since the same number of pieces were left over. When asked which is larger, 2/5 or 3/7, some of the students reasoned that when you use sevenths there are more pieces and hence the pieces are smaller; so 3/7 is smaller than 2/5. The students were seemingly unmindful of the role of the numerators.

Students who were adept at comparing fractions conceptually used some interesting comparison strategies for certain fractions; e.g. they reasoned that 2/5 is greater than 1/4 by pointing out that the 3/4 left over from the 1/4 is larger than the 3/5 left over from the 2/5, so 2/5 is greater than 1/4. This is sometimes called the "residual strategy", and it works only for those fractions whose residuals can be easily compared. Such a strategy would be categorized by Post et al. (1985) as a local strategy, that is, a strategy that is only a function of a specific task and does not necessarily persist or transfer to different situations.

In analyzing cognitively the issues involved in the determination of order and equivalence of fractions, Post et al. (1985) reiterate many of the reasons why the understandings about fractions are so complex (p. 33).

1. Fraction size depends on the relation between the two whole numbers in the fraction symbol.

2. There is an inverse relation between the number of parts into which the whole is divided and the resulting size of each part.

3. When fractions have like denominators, there is a direct relation between the number of parts being considered (numerators) and the order of the fractions.

4. When fractions have like numerators, there is an inverse relation between the denominators and the order of the fractions.

5. When fractions have different numerators and denominators, judgments about the order require an extensive and flexible use of fraction equivalence.

6. The density of rational numbers implies a counter-intuitive notion that there is no "next" fraction.

They also point out some very important linguistic considerations that contribute to children's misunderstandings. "More or greater" can mean to a child more area covered by each part. In fact, this very confusion as to whether "greater" means a greater number of parts in the partitioned whole or a greater fraction size came up in the May 1985 pilot study.

Both the model for a clinical teaching experiment and the analysis of the cognitive issues related to order and equivalence offered by the members of the Rational Numbers Project will be useful in the work of the ETC

Fractions Group. It will be interesting to see if our combination of the use of hands-on experiences and microcomputer software will result in many of the same insights into children's strategies and understandings, and to see what further thought processes and solution strategies children may develop and use when provided with the various pieces of software in Stage 4.

Summary

The major features of the Stages 1 - 4 activities that set them apart from other research efforts are:

1. The use of purely a linear continuous model.
   (Students do not use any pies, nor regions cut in more than one direction, nor discrete sets.)

2. The use of fractions in a measurement context.
   (Students see the need for fractions in order to obtain more exact values for lengths of continuous quantities.)

3. The use of both whole-to-part and part-to-whole relationships.
   (Students use both subdivision and iteration techniques in building rulers.)

4. The merging of the whole-to-part and part-to-whole subconstruct with the measurement subconstruct.
   (Students use their folded strips to make number lines subdivided into the various equal parts, halves, thirds, fourths, etc.; label the values at the end of the subdivisions in a continuous fashion, not internal to the interval; and then use the number lines as rulers for actual measurements.)

5. The labeling of the number lines (rulers) in actual subdivisions.
   (Students are not asked to name points on "doubly subdivided" number lines; for example, they would not label 4/6 as 2/3 but rather see the equivalent values by vertically eyeballing an array of carefully placed number lines.)

6. The active use of number lines (rulers) to show not only equivalence, but also order and betweenness.
   (Students use arrays of number lines to move horizontally in an additive fashion for fractions with the same denominator and to view vertically the interrelationships between fractions of different denominators. The emphasis is on active use of number lines, not just the static placement of values.)

7. The extension of lengths beyond one in a natural continuous fashion.
   (Students see the need for rulers longer than one whole unit in Stage 1 even before fractions are introduced.)

8. The use of technology merged with and to follow the manipulative tasks.
   (Students work first with hands-on experiences and then are given similar but more involved tasks on the computer; for example in the Island Hopping Games, Numberlines Environment, and the Mixed Number Measurement Activities.)

9. The interest in the notion of betweenness as well as the concepts of order and equivalence.
   (In the context of measurement, students see lengths as lying between two values, one too small and one too large. Students are asked to find closer and closer measurements of lengths and in so doing actively deal with both the issues of order and betweenness of fractions.)

10. The opportunity to build strategies for determining order and betweenness relationships.
    (The measurement activities and the computer games, especially those in Stage 4, help students evolve strategies as they discuss why they would choose certain fractions. The emphasis is on a visual sense of continuous quantity rather than on abstract rules involving merely the maneuvering of symbols.)

THE DEVELOPMENT OF TEACHING MATERIALS

As part of the planning of the teaching experiment, the ETC Fractions Group also reviewed much of the available curriculum material dealing with order properties of fractions and betweenness. It was found that most of the available traditional materials - textbooks, workbooks, manipulatives, etc.- did not support our emphasis on equivalence, ordering and betweenness of fractions. Furthermore, the presentation of fractions in commercial materials tended to utilize a variety of models, few of which focus on the model of continuous quantity and measurement.

With the intention of integrating educational technology into the project, the group reviewed existing fraction software. Most of what is available either deals with the order properties in terms of discrete referents or abstractly, in terms of no referents at all. In contrast, the group's working hypothesis depends on the development of children's understanding of the order properties in the context of continuous referents.

Much of the existing fraction software that does deal with the quantification of continuous quantity does not distinguish clearly between one dimensional and more-than-one dimensional referents. This presents a particular difficulty to the working group because of the centrality of the notion of betweenness to its work. Betweenness is a property of a one-dimensional referent only. Additionally, it was found that most of the available software that deals with the order properties of fractions does so in ways that are deeply embedded in game contexts that were judged by the practitioners to be so distracting in the use of irrelevant graphics and sound as to lose most if not all of their usefulness.

There were, however, two pieces of commercial software which the group felt would be useful in the teaching experiment at Stage 4 involving the ordering and betweenness properties of fractions. The first was the fractions portion of Number Quest published by Sunburst; the second was Darts, published by Apple Computer, Inc. Both were used in the pilot study conducted in May 1985 and are described under Stage 4 activities. These two programs will be studied in more depth as part of the research project to be conducted during the academic year 1985-1986.

The task then became to design and develop the majority of the materials used in the research project. Drawing from the literature review, the curriculum materials review, the software review, and the collective teaching experience of group members, activities and materials were designed to be developmentally appropriate based on our theoretical framework. The materials and activities were also designed to provide a range of experiences for the students - paper and pencil, manipulative, software, etc.

The ETC Fractions Group has planned and designed three new pieces of software, each in the process of development at the present time. Our focus is not on how to integrate technology into the learning of fractions, but rather what is it about an understanding of equivalence, ordering and betweenness of fractions that can be fostered using the technology. The main purpose of each program is to provide an experience for students that could

not be possible without the use of the computer. Most of the ideas for software evolved from successful paper and pencil activites which the group felt could be extended and enhanced through the computer.

Preliminary versions of these programs were available for the May 1985 pilot study. The feedback from this initial use will lead to extensions and revisions. The Island Hopping Computer Games proved to be highly successful with the students as part of the Stage 2 activities and will be extended to be included also in Stage 4. It goes far beyond the hands-on Island Hopping Games by including more difficult denominators, more challenging game options, and more decision-making about the ordering and betweenness properties of fractions.

The Numberlines Computer Games will be revised considerably to focus more on the equivalence and ordering properties of fractions with different denominators than on the sequencing aspects of the fractions with the same denominators. The May 1985 pilot study provided the impetus for the ETC Fractions Group to develop a computerized numberlines environment which will provide opportunities for students to visualize order and betweenness relationships in ways that would be impossible with paper and pencil or paper folding techniques.

The Mixed Number Measurement Software described and illustrated under Stage 3 activities, was used in both the December 1984 and May 1985 pilot studies. Consequently it has gone through several revisions and is the most highly perfected of the three pieces of software being developed by the ETC Fractions Group. This software allows students to use all denominators through tenths and to be able to make more and more precise measurements. Students can immediately see the consequences of their selection of a denominator to appreciate the ordering and betweenness relationships in the practical setting of measuring lengths.

The working group is studying the potential of interactive software to provide experiences that would be difficult or impossible to accomplish with other materials. The two commercial programs and the three pieces of software under development by the ETC Fractions Group will be used in conjunction with the Stages 1 - 3 hands-on activities to help children gain a better understanding of the order and betweenness properties of fractions and the use of fractions as measurement quantities.

Both university-based and school-based members of the group have worked in consultation with software designers and programmers. Our aim has been to build the software on our knowledge of theory, practice, and technical potential and to revise it in light of students' responses.

Testing the activities (in the December and May pilot studies) while they were being developed helped the group to judge the effectiveness of each individual task and to sequence from task to task, and allowed for the development of ideas for the interactive pieces of software vital to the research.

Outline of the Stages 1 - 4 Activities

Stage 1 Simple Length Measurement: Integer Number of Units

Worksheets 1.1 and 1.2

Stage 2 Length Measurement: Non-Integer (Less Than One) Number of Units

Measuring Strips Activities
Worksheets 2.1 - 2.4
Projection Activities (Optional)
Worksheet 2.5
Island Hopping Games
Island Hopping Computer Games

Stage 3 Length Measurement: Non-Integer (Greater Than One) Number of Units

Part - Whole Relationships Activities
Worksheets 3.1 and 3.2
Numberlines Mat Activities and Games
Numberlines Computer Games
Mixed Number Measurement Activities
Worksheets 3.3 and 3.4
Mixed Number Measurement Software

Stage 4 Order Relationships of Fractions and the Notion of Betweenness

Darts
Number Quest

Descriptions of the Stages 1 - 4 Activities

Stage 1 - Simple Length Measurement: Integer Number of Units
Worksheets 1.1 and 1.2

The purpose of this part of the teaching procedure is to emphasize the notion that the numerical magnitude of continuous quantity is dependent on the unit that is used.

Materials:

Box of pencils
Box of straws
Orange Cuisenaire rods (or strips of paper approx. 10 cm X 1 cm)
Colored strips of paper 10 in X 1 1/2 in
Worksheets 1.1 and 1.2
Roll of tape from an adding machine
or roll of masking tape

Worksheet 1.1

Hand out copies of Worksheet 1.1.
Choose an object in the room of which several exist that are the same size so that many students can do the same measurement activity at once (for example: tables, window sills, student desks, or sections of a blackboard).
Each student estimates how many pencils would be needed to measure the length of the chosen object. The student records the estimate in the proper place on Worksheet 1.1.
The teacher demonstrates measuring the length of that object by laying down pencils end-to-end.
Each student then records the measurement in the proper place on Worksheet 1.1 as being between ____ and _____ pencils.

Students work together in groups to do a similiar measuring process of the same object using different units of measure (straws, Cuisenaire rods, and colored strips of paper). Each student draws around the unit of measure in the space provided on Worksheet 1.1 and records his or her estimates and measurements. (The pencil and colored strip of paper have been drawn for them.)

Compare the estimates and measurements.
Discuss the relationship between the relative size of the unit of measure and the number of units needed for the given length. (e.g. The longer the unit of measure, the fewer the number of units needed.)

Stage 1 - Worksheet 1.2

Students estimate the height of the blackboard in pencils and call out their guesses for the teacher to record in a list on the blackboard.
The teacher demonstrates how to measure the height of the blackboard by using the process of iteration since it is not practical to measure height using multiple copies of the unit of measure. The teacher shows how to use one pencil to measure by repeatedly putting the pencil down and using a finger to designate the end of the pencil.

Students estimate the height of a particular student in terms of a pencil as a unit of measure.
As a demonstration, one student measures another using a pencil and the iteration method.
Discuss the difficulties in having to iterate the unit of measure when measuring heights.

The teacher then demonstrates an alternative method, constructing a measuring tape with marked iterations of the pencil. The teacher marks off "pencil units" on adding machine tape or masking tape (a length of 6 - 7 feet). The teacher holds the measuring tape up to the blackboard and measures the height of the blackboard using the constructed measuring tape.

Attach the measuring tape to a wall. Measure the same student whose height was determined by the iteration method.

In small groups the students construct measuring tapes using Cuisenaire rods, straws, and paper strips as units of measure. (Each group makes one type of measuring tape so that there are two of each kind available for class use.)
Attach each measuring tape to a wall.

Hand out Worksheet 1.2.
Students draw around the unit of measure in the space provided on Worksheet 1.2 and then estimate their own heights in terms of each unit of measure and record those estimates on Worksheet 1.2. (The pencil and colored strip of paper have been drawn for them.)
Students work in pairs to measure each other's heights using the measuring tapes and record their own measurements in the proper places on Worksheet 1.2.

Discuss the accuracy of each method - iteration vs. using a measuring tape.
Discuss the notion again of the inverse relationship between the size and the number of units of measure.
Discuss the need for fractions as "numbers between numbers."

OBJECT MEASURED _____

UNIT OF MEASUREMENT	ESTIMATE	MEASUREMENT
pencil	____ pencils	between ____ and ____ pencils
_____	____	between ____ and ____
_____	____	between ____ and ____
paper strip	____ strips	between ____ and ____ strips

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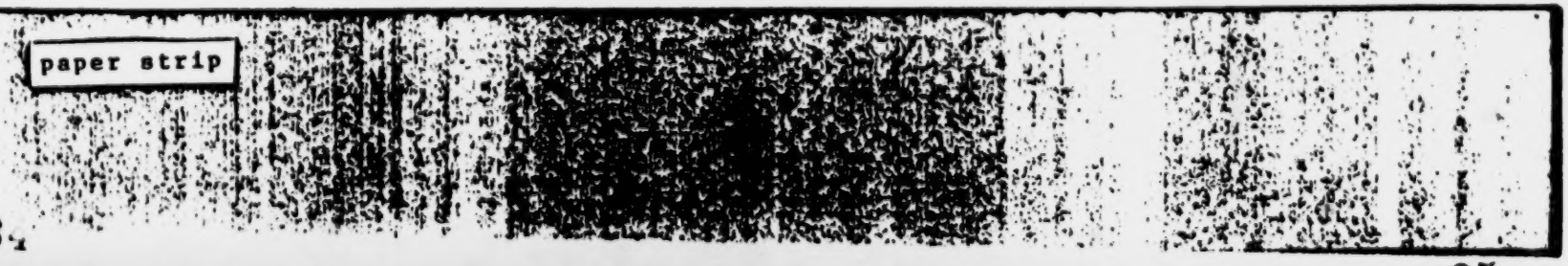
OBJECT MEASURED _____

MEASURING TAPE UNITS	ESTIMATE	MEASUREMENT
pencil	____ pencils	between ___ and ___ pencils
_____	____ ____	between ___ and ___ _____
_____	____ ____	between ___ and ___ _____
	____ strips	between ___ and ___ strips

Stage 2 Length Measurement: Non-Integer (Less Than One) Number of Units

Stage 2 - Measuring Strip Activities

Students make rulers with half, third, fourth, sixth, and eighth subdivisions by folding paper strips.

Materials:

Strips of colored construction paper (approximately 10" X 1 1/2") with a black line across the top edge - enough strips so that each student has two of each color plus extras (particularly of yellow and red)
- green - to be folded into halves
- yellow - to be folded into thirds
- blue - to be folded into fourths
- red - to be folded into sixths
- purple - to be folded into eighths

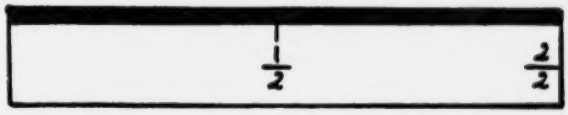
Scissors
Pencils

Activities:

Fold two green strips, each into 2 equal parts.

Fold two blue strips, each into 4 equal parts.
Fold two purple strips, each into 8 equal parts.

Cut one of the green strips into the two parts.
Lay one green part on the other whole green strip.
Put a mark on the whole green strip where the green part ends.
Label that mark as 1/2 on the whole strip. Then put the second green part down and mark that 2/2.
Discuss the terms "numerator" and "denominator."



Continue the cutting, marking and labeling for the strips - for fourths (1/4, 2/4, 3/4, 4/4) and for eighths (1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8).

Fold two yellow strips, each into 3 equal parts.
Fold two red strips, each into 6 equal parts.
Continue the cutting, marking and labeling for the strips - for thirds (1/3, 2/3, 3/3) and for sixths (1/6, 2/6, 3/6, 4/6, 5/6, 6/6).

Stage 2 - Worksheets 2.1 - 2.4

Materials:

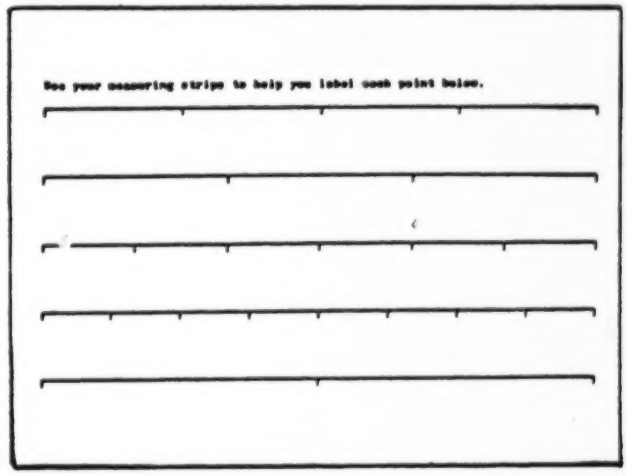
Measuring strips
Thin pencils or felt pens
Worksheets 2.1, 2.2, 2.3, and 2.4

Worksheet 2.1

Strips are used by students to label points on unit number lines.

Worksheet 2.1

On Worksheet 2.1, label the lines like the measuring strips. Observe the marks at the beginning of each line and label them appropriately with 0/4, 0/3, etc. Use measuring strips if needed.

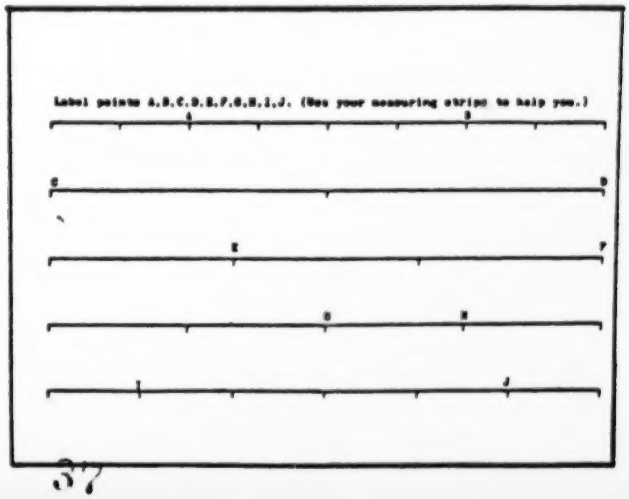


Worksheet 2.2

As a variation of the first worksheet, students are asked to select an appropriate ruler (measuring strip) of their own making and label the fractional parts of a unit distance which are marked with letters. All fractional subdivisions on the line are given.

Worksheet 2.2

On Worksheet 2.2, label the particular fractional lengths which are marked with letters. All fractional subdivisions are marked on the line.
Use measuring strips if needed.



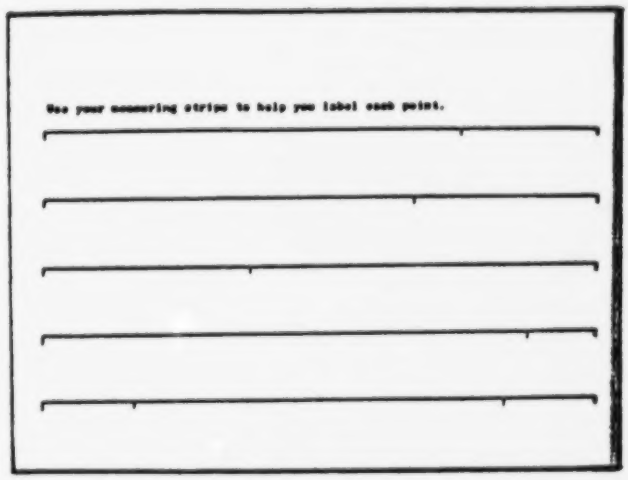
Stage 2 - Worksheets 2.3 and 2.4

Worksheet 2.3

As still another variant, students are asked to select an appropriate ruler of their own and label particular fractional lengths of a unit length where no other fractional subdivisions are marked on the line.

Worksheet 2.3

On Worksheet 2.3, label particular fractional lengths on lines where no other fractional subdivisions are marked.
Use measuring strips to choose appropriate denominators.

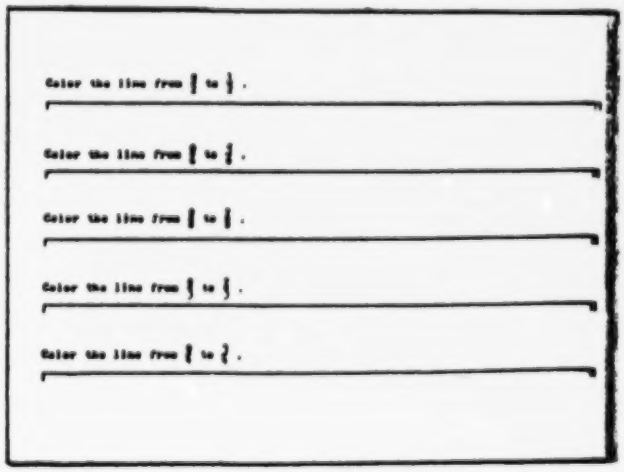


Worksheet 2.4

Students are then asked to mark off given line segments on a unit number line using the paper rulers they have made.

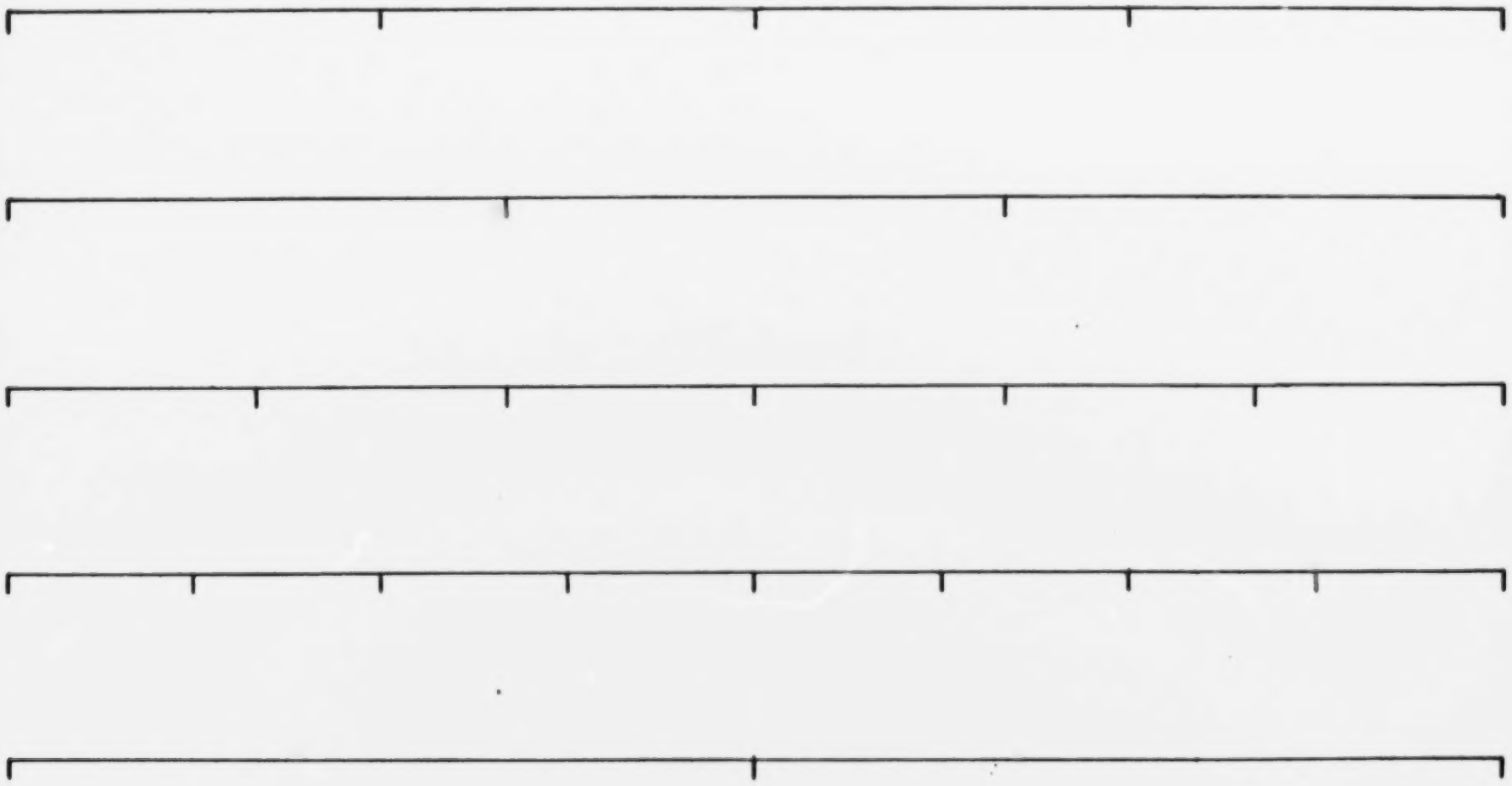
Worksheet 2.4

On Worksheet 2.4, draw the particular fractional lengths.
Use thin colored pencils or felt pens and the appropriate measuring strips.

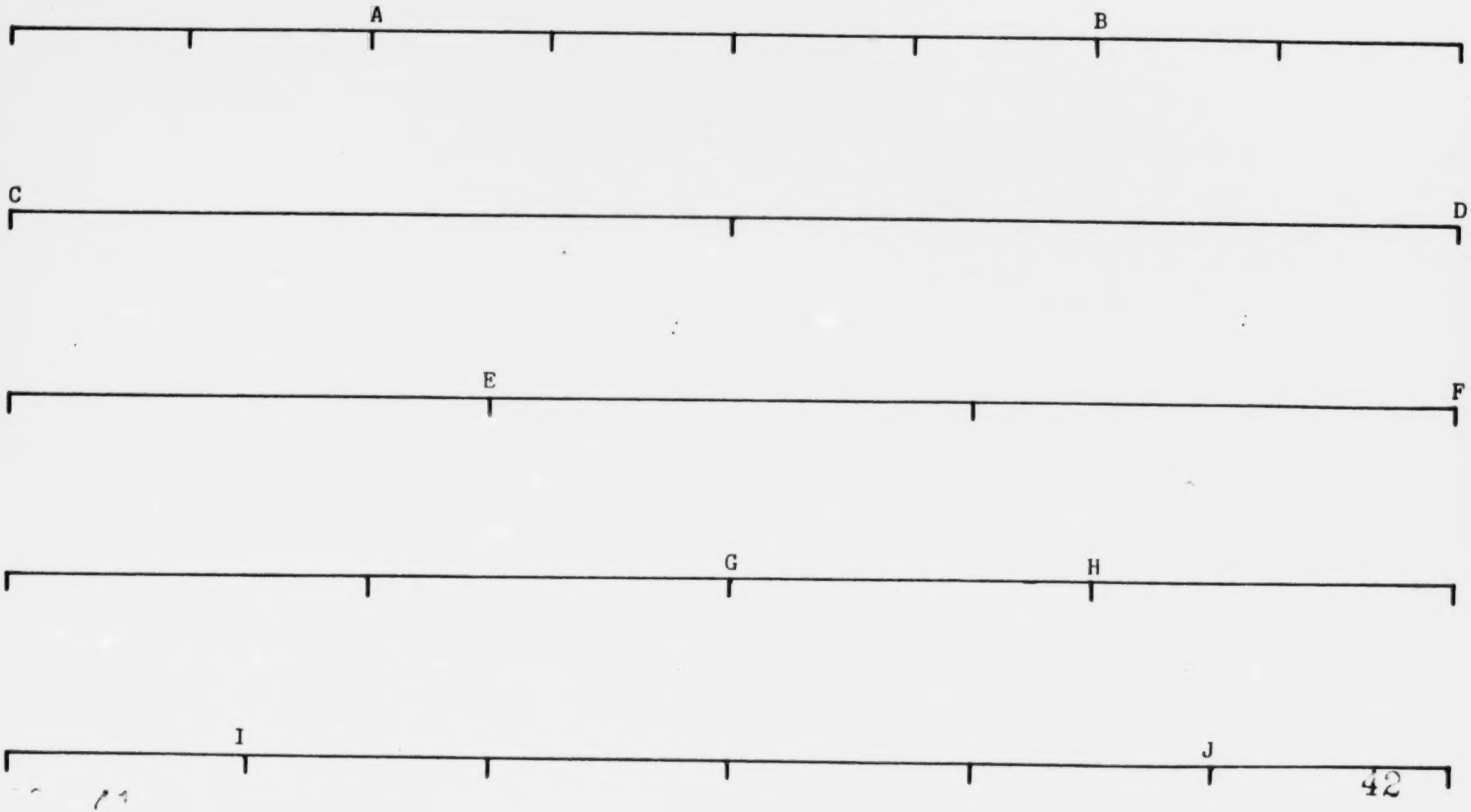

Color the line from 0 to 1/3.
Color the line from 0 to 2/3.
Color the line from 0 to 1/4.
Color the line from 0 to 3/4.
Color the line from 0 to 2/4.

Use your measuring strips to help you label each point below.

Worksheet 2.1



Label points A, B, C, D, E, F, G, H, I, J. (Use your measuring strips to help you.) Worksheet 2.2



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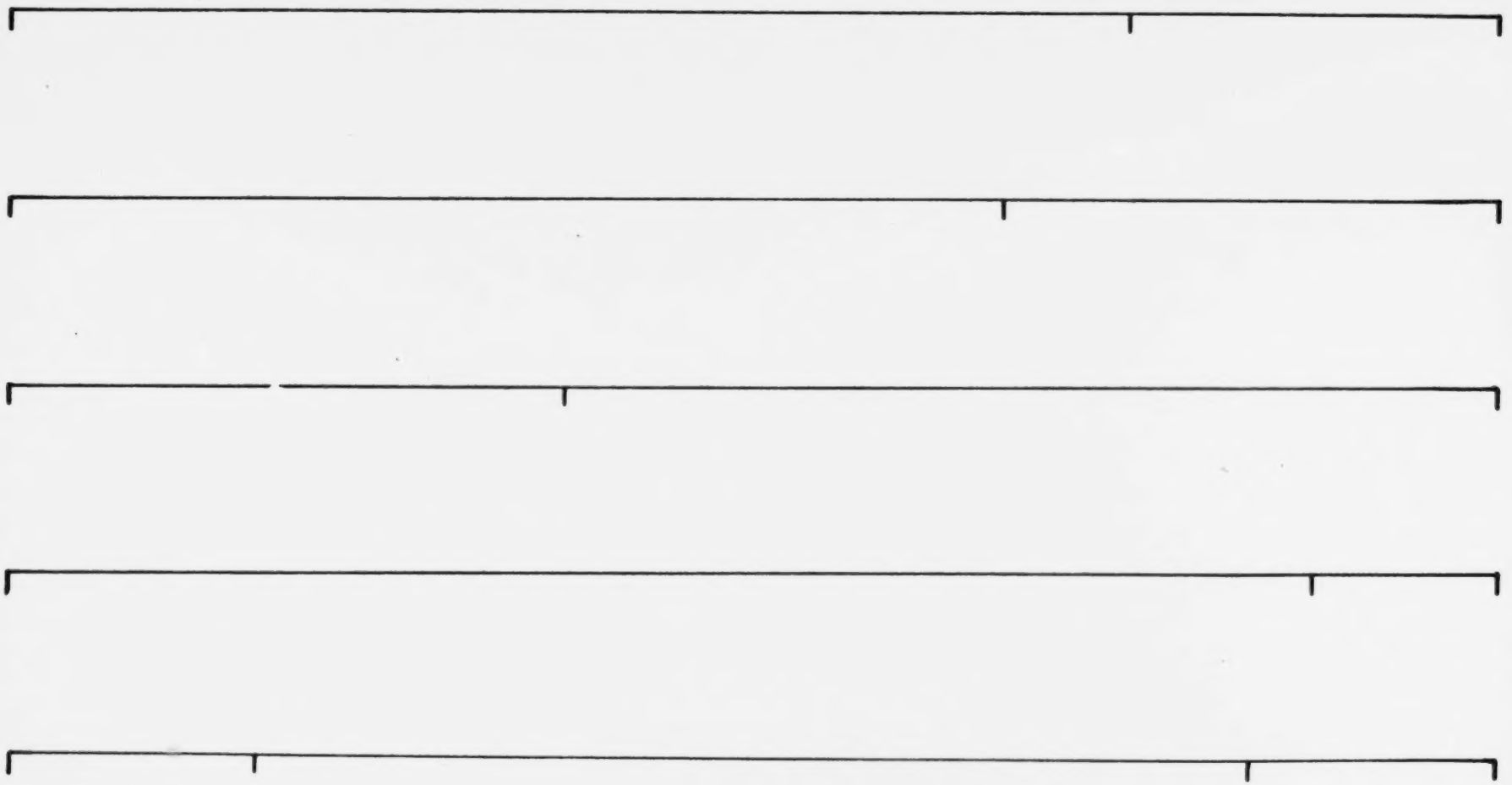
Use your measuring strips to help you label each point.

Worksheet 2.3



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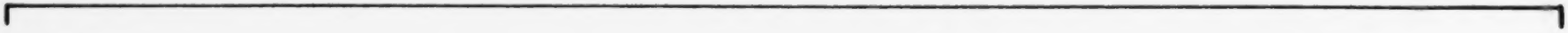
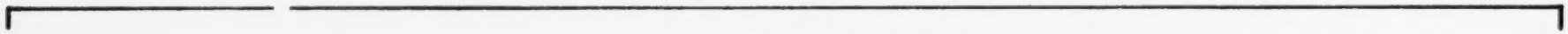
Color the line from $\frac{0}{2}$ to $\frac{1}{2}$.

Worksheet 2.4



Color the line from $\frac{0}{6}$ to $\frac{5}{6}$.



Color the line from $\frac{0}{8}$ to $\frac{2}{8}$.

Color the line from $\frac{0}{3}$ to $\frac{2}{3}$.

Color the line from $\frac{0}{4}$ to $\frac{3}{4}$.

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Stage 2 - Projection Activities (Optional)
Worksheet 2.5

After completing the paper folding activities to obtain halves, thirds, fourths, sixths and eighths, students may be shown a projection technique. This technique can be used to obtain other fractional parts of a given unit of measure, for example, fifths, sevenths, ninths, or tenths, for which paper folding techniques are not feasible.

Materials:

- Sample Projection for Five Equal Parts
- Worksheet 2.5 (5 or 6 per student)
- Straightedge for drawing straight lines
- Pencil

The Projection Worksheet is set up so that students can subdivide the same length as the paper strips they have been using in the paper folding activities. Hence, the same paper strip is pictured at the bottom of the worksheet. Above the paper strip is a line drawn parallel to the paper strip and subdivided into equal parts. (The subdivisions of the line on the worksheet are inches and half inches, but any equally spaced subdivisions would work.)

Decide on how many equal parts (n) into which you want to subdivide the paper strip. Then label equally spaced subdivisions on the line from 0 to n. Choose the subdivisions in the center of the line rather than at the extreme ends so that the projection point (described in the next paragraph) will not land off the paper. (See sample projection for five equal parts.)

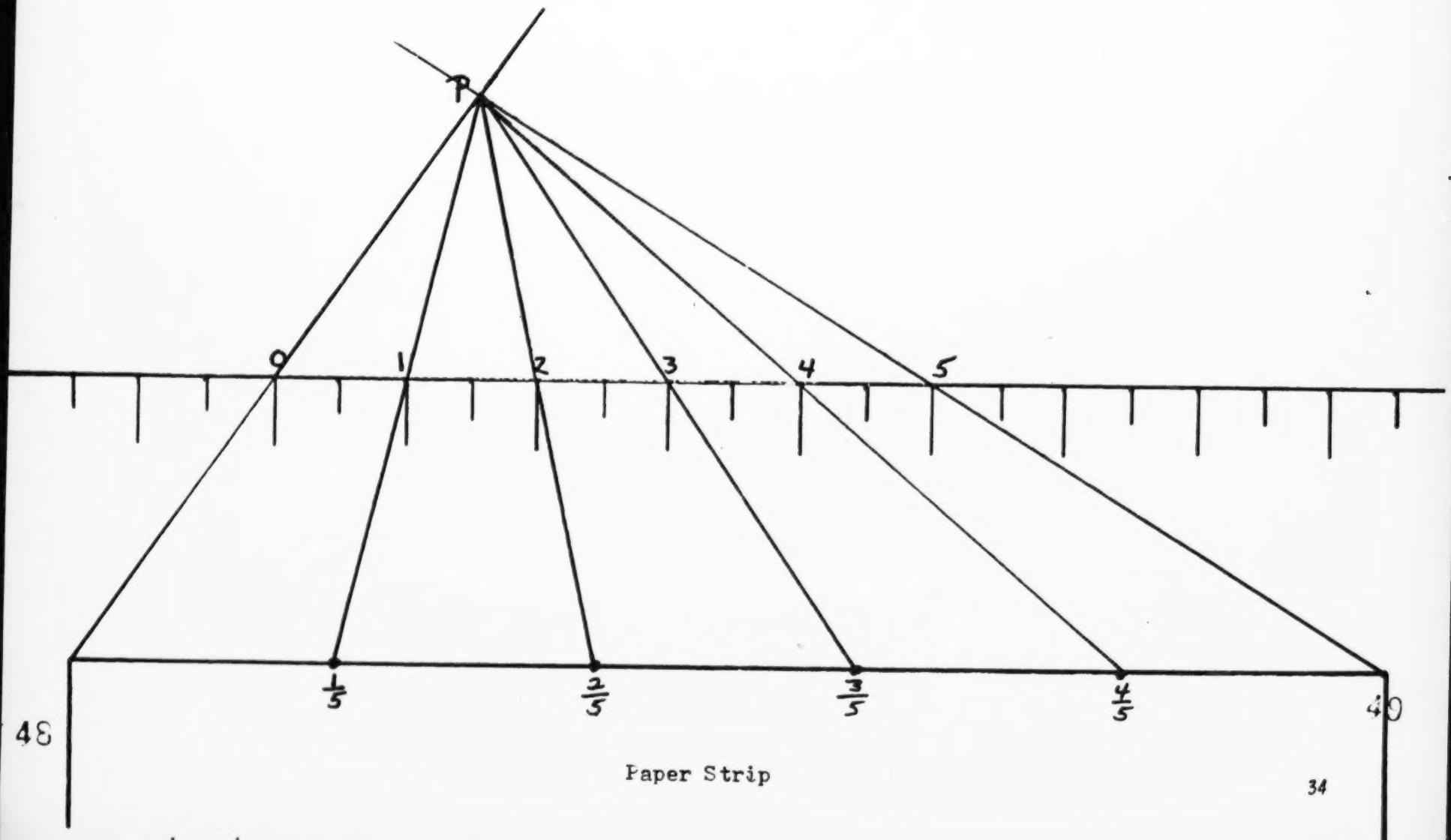
Draw a straight line from the upper left hand corner of the paper strip to the point on the line labeled 0. Do the same from the upper right hand corner of the paper strip with the point on the line labeled by the number n. Label the point where these two lines meet as P (projection point).

Now connect P to the point on the line labeled 1 and continue drawing until the line intersects the paper strip. Label this point 1/n. Then connect P to the point on the line labeled 2 and continue drawing until the line intersects the paper strip. Label this second point 2/n. Continue for the rest of the labeled points 3, 4, ... n. The left hand corner of the paper strip should then be marked 0/n and the right hand corner n/n. Now the paper strip can serve as a ruler subdivided into n equal parts.

The students can use this technique to make several different fractional rulers (fifths, sevenths, ninths, tenths). They should compare their work with other classmates' work to see that it does not matter where they placed the 0 or whether they used the small or large subdivision markings on the parallel line provided they labeled equal subdivisions from 0 to n.

Students may also wish to use the projection technique for halves, thirds, fourths, sixths or eighths to see that they get the same subdivisions as they obtained when using the paper folding technique.

SAMPLE PROJECTION FOR FIVE EQUAL PARTS



Paper Strip

PROJECTION WORKSHEET 2.5

Paper Strip

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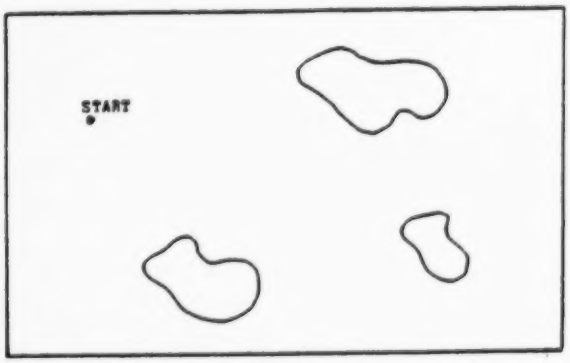
Stage 2 - Island Hopping Games

The next activity involves a game that requires students to plot a course through a collection of islands. The size of each move is determined by a fraction card that the student draws. Each card is marked with a fraction referring to a segment of the measuring strip unit. The students use the measuring strips that they made to draw the lengths of their moves. Each student has an island map. The winner of the game is the first player to pass in and out of each island exactly once and then through the START again.

Materials:

Island maps (on 11in X 14in paper) - one for each student
Measuring strips
Deck of fraction cards
Pencils

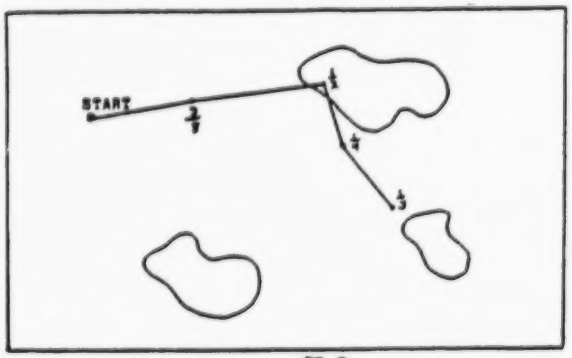


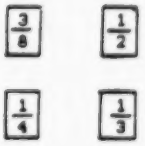
Placed faced down is a shuffled deck of fraction cards with these fractions:

0/2, 1/2, 2/2, 0/3, 1/3, 2/3, 3/3, 0/4, 1/4, 2/4, 3/4, 4/4, 0/6, 1/6, 2/6, 3/6, 4/6, 5/6, 6/6, 0/8, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, and 8/8.

On each turn, a player picks or is dealt a fraction card and then, using the appropriate measuring strip, draws and labels that fractional length. When subsequent cards are picked or dealt, the players continue drawing on from the end of their previous fractional length.

An example of the first four moves in a game might look like this:





Stage 2 - Island Hopping Game (continued)

The following information should be communicated to the students:

1. Players can think of themselves traveling in an amphibious vehicle.

2. Players can go to the islands in any order and must pass in and out of each island.

3. Players must actually stop at each island. (The end of the fractional length must be within the boundaries of the island.)

4. Players may not be able to stop at an island, but are still able to draw the fractional lengths and hence may spend a considerable number of turns "treading water."

5. If a length is too long and requires either entering an island again or going off the paper, the turn will be lost for that player.

6. The fractions 0/2, 0/3, 0/4, 0/6, and 0/8 designate lengths of "0" or "stay put."

7. Correctness of the students' drawn lengths should be checked.

8. Players should compare strategies.

Island Hopping Game: Five Versions

1. Only one fraction card is taken for each turn and is used by all players.

2. A fraction card is dealt to each player, and the group waits until all players have drawn and labeled their fractional lengths before new cards are dealt.

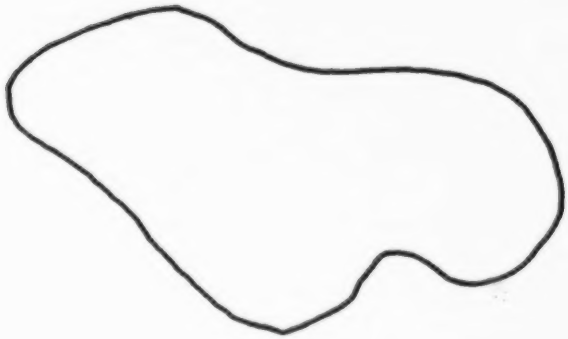
3. As many fraction cards as players are drawn and given to one of the players who gets first choice. The remaining cards are passed around and the players take turns selecting which fraction they will use. The player who gets the first choice rotates on each round.

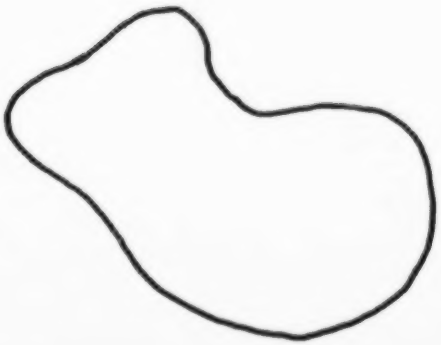
4. Five fraction cards are dealt to each player who may use them in any order. If no player has completed the trip in the first five moves, five more cards are dealt to each player. The player who completes the trip using the smallest number of fractions wins.

5. Five fraction cards are dealt to each player. Each player must write down the order in which he or she plans to use the cards before beginning to draw the route. If no player has completed the trip in the first five moves, five more cards are dealt to each player. The player who completes the trip using the smallest number of fractions wins.

ISLAND HOPPING GAME

START





Stage 2 - Island Hopping Computer Games

The Island Hopping Games on the computer are extensions of the paper games previously described. Students travel to islands by using fractional lengths but play on the same screen. The computer games offer different kinds of options for the players.

Materials:

IBM PCjr or PC with graphics board
Island Hopping Game disk

Students can invent games from the different options built into the software. Some options available to them are:

Choose 2-4 islands (pre-placed)

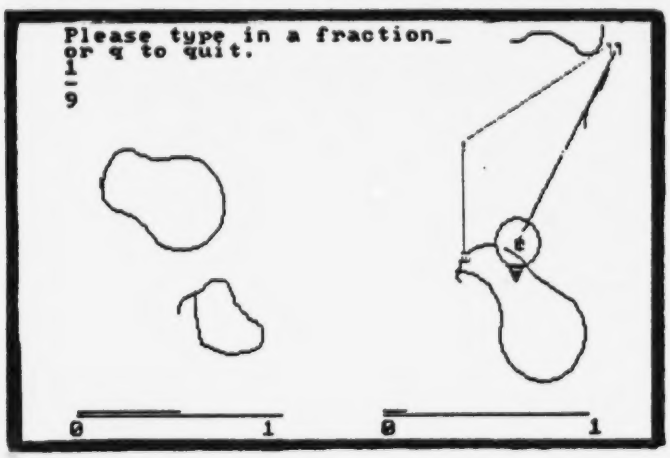
Choose 1-5 random fractions or players can input their own fractions

Choose from two groups of fractions: 1) values between 0 and 1 in halves, thirds, fourths, sixths, and eighths, and 2) values between 0 and 1 in fifths, sevenths, ninths and tenths as well.

Choose 1 or 2 routes which are presented in two different colors

Choose one unit of length or two different units of lengths with adjusted fractions. (e.g. if unit A is twice unit B, fractions for player A will be 0 to 1/2 while fractions for player B will be 1/2 to 1.)

Once the student chooses a fraction, a circle with that length as the radius appears on the screen with the center of the circle at the point where the player is to start that turn. The direction the player will move is then determined by the player moving a cursor around the circle to the desired end-point for the move.



Stage 2 - Island Hopping Computer Games (continued)

Some possible games:

1. Each player has own route. Players take turns. They can pick one of two fractions shown on the screen. Two new fractions appear each turn.

2. Each player has own route. Players pick from the same bank of five fractions shown on the screen. Each time a fraction is used, another replaces it.

3. One route to three or more islands is shared by two players. The first player types in a fraction for the distance and determines the direction of the first move. After that, the players take turns typing the choices determining the fraction and direction. The purpose is to reach an island on your turn. No player may get to an island in less than three turns. The player who "captures" the most islands wins.

Stage 3 Length Measurement: Non-Integer (Greater Than One) Number of Units

Stage 3 - Part - Whole Relationships Activities
Worksheet 3.1 and 3.2

Previously students have been given a whole unit of measure and asked to find or label the fractional parts. In this activity, students are asked to reverse the process and go from parts to the whole. They will be given a fractional length and will be asked to find the whole or another fractional length. They will use Cuisenaire rods to help them.

Materials:

Worksheets 3.1 and 3.2
Cuisenaire rods
Pencils

Worksheet 3.1

In parts 1a-c, students are given a fractional length designated with a fraction with 1 in the numerator. They will find a Cuisenaire rod equal to that given fractional length and will place the appropriate number of rods end-to-end.

For example, in 1a, the rod that matches 1/4 is a purple rod. To find the whole, the students will place 4 purple rods end-to-end, draw that length on the designated line, and label the right most end point 4/4.

In parts 2a-c, students are again given a fractional length designated with a fraction with 1 in the numerator. This time instead of finding the whole, they are asked to draw another fractional part. Students will use Cuisenaire rods to help them.

For example, in 2a, the rod that matches 1/5 is a green rod. To find 4/5, the students will place 4 green rods end-to-end and draw that length on the designated line.

Stage 3 - Worksheet 3.2

In parts 1a-c, students are given a fractional length designated with a fraction whose numerator is not 1. Students will use Cuisenaire rods to find the length representing the fraction with a numerator of 1 and then will place the appropriate number of rods end-to-end to produce the whole.

In 1a, 3 purple rods represent 3/4; hence 1 purple rod represents 1/4 and 4 purple rods will represent the whole.

In 1b, 5 red rods represent 5/8; hence 1 red rod represents 1/8 and 8 red rods will represent the whole.

In 1c, 4 red rods represent 4/7; hence 1 red rod represents 1/7 and 7 red rods will represent the whole.

In parts 2a-c, students are given a fractional length again designated with a fraction whose numerator is not 1. Students will use Cuisenaire rods to find the length representing the fraction with numerator 1 and then will place the appropriate number of rods end-to-end to produce the desired fractional length.

In 2a, 3 green rods represent 3/5; hence 1 green rod represents 1/5 and 4 green rods represent 4/5.

In 2b, 7 white rods represent 7/11; hence 1 white rod represents 1/11 and 9 white rods represent 9/11.

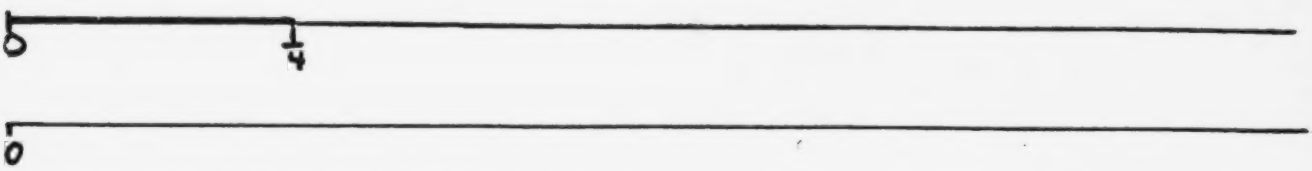
In 2c, 4 red rods represent 4/9; hence 1 red rod represents 1/9, and 6 red rods represents 6/9 which is equivalent to 2/3.

Another way to do this one is to build the whole with 9 red rods. To find 2/3, students can see that 1 dark green rod is 1/3 of the whole, so 2 dark green rods would be 2/3. Since 2 dark green rods are equivalent in length to 6 red rods, either method can be used to draw the length 2/3.

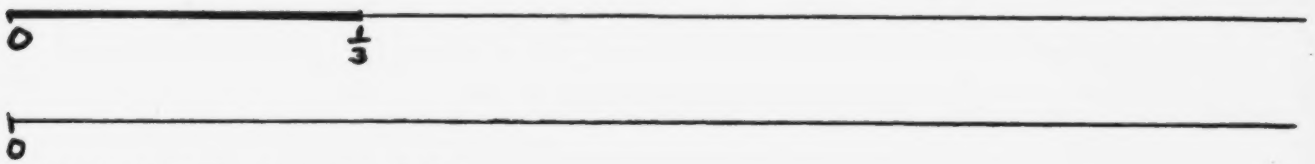
WORKSHEET 3.1

1. Use rods to help draw the whole on the line below each part.

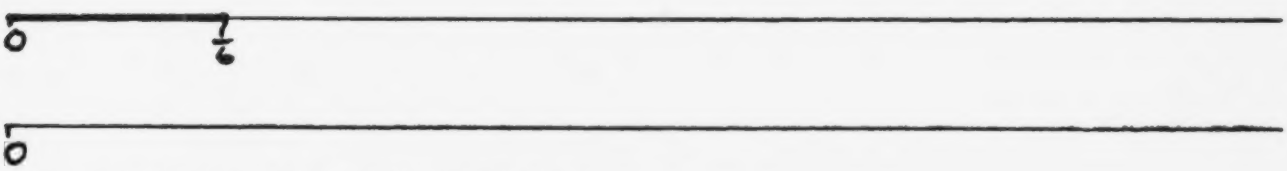
a. If this is 1/4, draw one whole.



b. If this is 1/3, draw one whole.

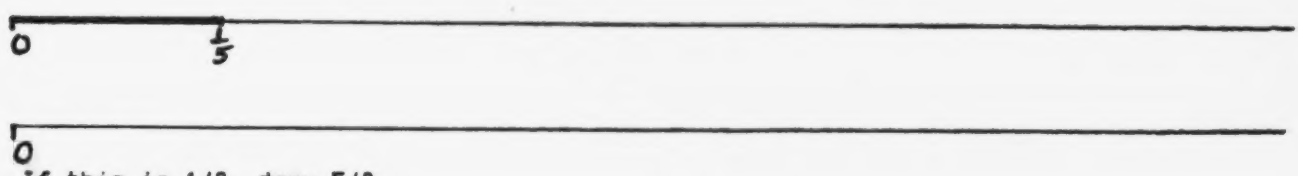


c. If this is 1/6, draw one whole.

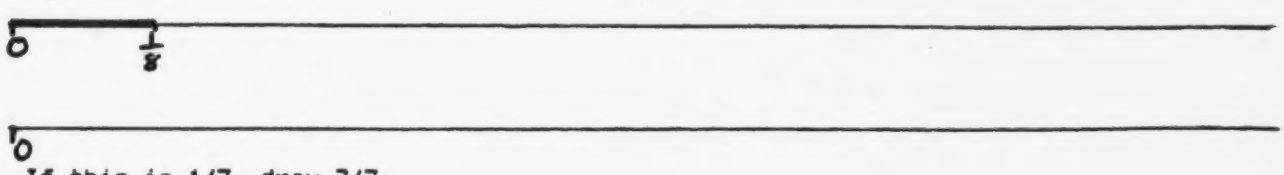


2. Use rods to draw the length asked for.

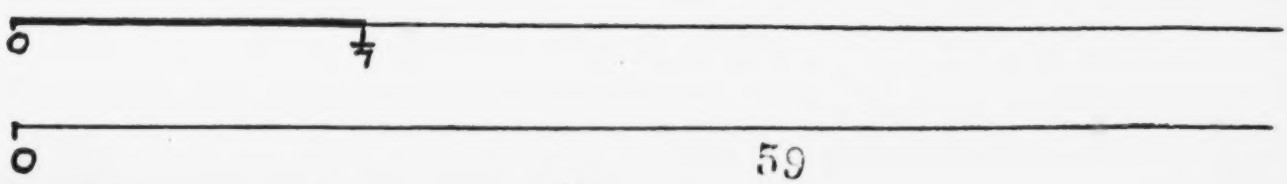
a. If this is 1/5, draw 4/5.



b. If this is 1/8, draw 5/8.



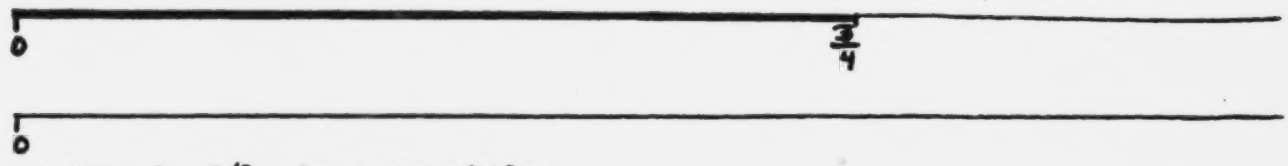
c. If this is 1/7, draw 3/7.



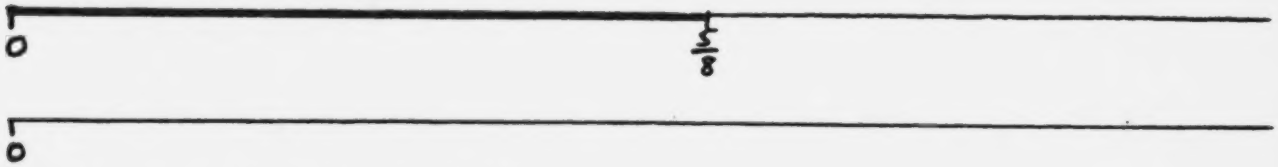
WORKSHEET 3.2

1. In each of these a part is given. Use rods to draw one whole on the line below each part.

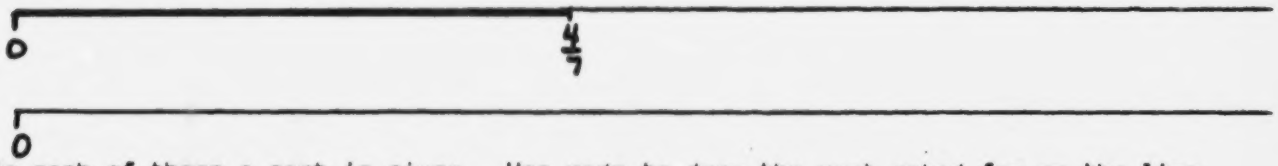
a. If this is 3/4, draw one whole.



b. If this is 5/8, draw one whole.

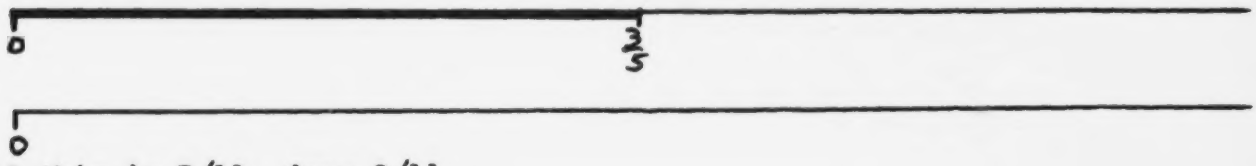


c. If this is 4/7, draw one whole.

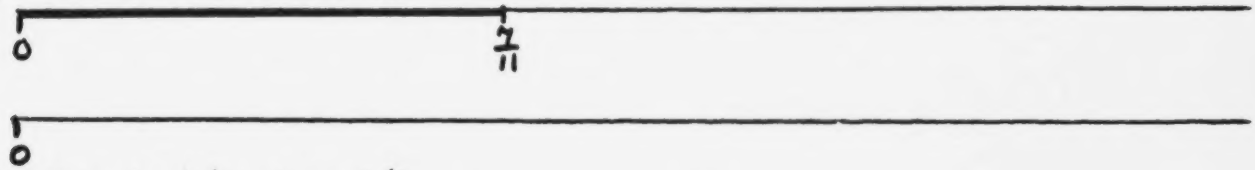


2) In each of these a part is given. Use rods to draw the part asked for on the line below.

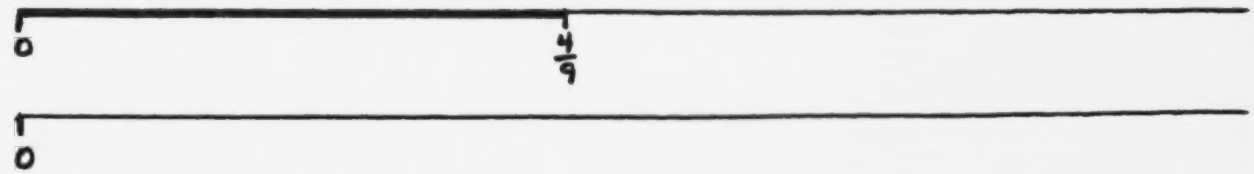
a. If this is 3/5, draw 4/5.



b. If this is 7/11, draw 9/11.



* c. If this is 4/9, draw 2/3.



* challenge for the pro!

Stage 3 - Numberlines Mat and Activities Games

The following numberlines mat activities and games are used to review and extend the previous work on equivalent fractions between 0 and 1, and to introduce improper fractions and mixed notation for fractions between 1 and 2. Since the mat is laid out so that the numberlines for halves, thirds, fourths, fifths, sixths, eighths, and tenths line up vertically, students can make visual comparisons among all the fractions pictured to determine order and betweenness relationships.

Materials:

A numberlines mat for each group of two or three students. (A reduced sample is enclosed.)

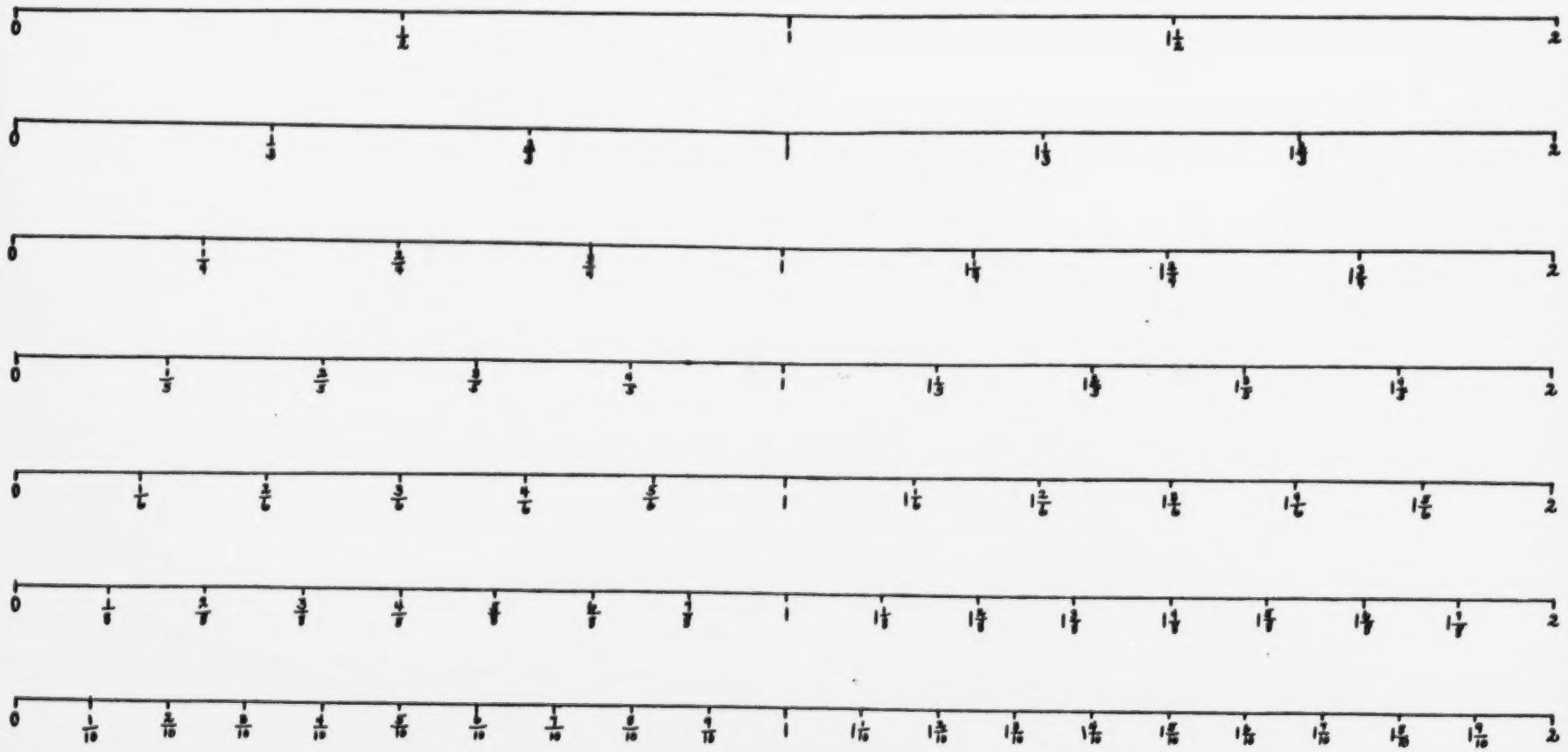
The mat should be drawn on a piece of cardboard approximately 21 in. X 14 in., and then laminated or covered with clear contact paper for durability. The unit of length (10 inches) should be used to be consistent with the fractional measuring strips used in previous activities. The values greater thatn 1 are labeled in mixed number notation.

Deck of fraction cards (halves, thirds, fourths, fifths, sixths, eighths, and tenths with proper and improper fractions less than or equal to 1 1/2) for each group of two or three students. The fractions on the cards are as follows:

1/2, 2/2, 3/2
1/3, 2/3, 3/3, 4/3
1/4, 2/4, 3/4, 4/4, 5/4, 6/4
1/5, 2/5, 3/5, 4/5, 5/5, 6/5, 7/5
1/6, 2/6, 3/6, 4/6, 5/6, 6/6, 7/6, 8/6, 9/6
1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8, 9/8, 10/8, 11/8, 12/8
1/10, 2/10, 3/10, 4/10, 5/10, 6/10, 7/10, 8/10, 9/10, 10/10, 11/10, 12/10, 13/10, 14/10, 15/10

Three colors of game markers, approximately 10 of each color for each group of two or three students

NUMBER LINES GAME



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Stage 3 - Numberlines Mat Activities

1. Students lay down their measuring strips along the appropriate lines of the numberlines mat to verify the relationship between improper fractions (obtained by counting up the fractional parts) and the mixed number notation used to label the points on the numberlines (e.g. six fourths (6/4) is equivalent to 1 2/4 on the fourths numberline).

2. Students look at the numberlines mat and call out equivalent fractions which are then listed on the blackboard.

3. Students are dealt a fraction card in turn from the deck. They have to find the location of that fraction, whole number, or mixed number on the numberlines mat and name all equivalent values on the mat.

4. Students use < signs to order from smallest to largest the fractions between 0 and 1 shown on the numberlines. They write equivalent fraction underneath each other in a column.

Example: $0 < \frac{1}{10} < \frac{1}{8} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} \ldots$

$\frac{2}{10} \quad \frac{2}{8}$

5. With eyes closed, students place their thumb and forefinger on the number lines mat. With eyes open, they move their thumb and forefinger to the closest marks on the numberlines mat. They name these values and also all the fractions, whole numbers, or mixed numbers between these two values. After some experience, students should name the values where their thumb and forefinger are located and then close their eyes again to tell at least two fractions, whole numbers, or mixed numbers between these two values.

Stage 3 - Numberlines Games: Five Versions

1. Seven markers of the same color are placed at the 0 on each of the number lines on the mat. The deck of cards is placed face down. On a turn a player takes the top card and moves that amount on any of the numberlines. For example, if the fraction card is 2/3, the player can move either a length of 2/3 on the thirds line or a length of 4/6 on the sixths line. The markers are common property and are moved along the lines from where the last play left off. If it is impossible for a player to move on any of the lines, the turn is lost (e.g if the marker were at 1 7/8 and 3/8 is drawn). The goal is to get to 2 exactly on any one of the lines so as to win a marker. Whenever a player wins a marker, a new marker is placed at 0 on that same numberline. The first player to win five markers wins the game.

2. The game is played as in version 1, except that the seven markers at the beginning of the game are placed at 2 on the number lines mat, and the goal is to reach 0 exactly. The first player to win five markers wins the game.

3. This version is for only for two players and is a combination of versions 1 and 2. One player places seven markers of one color at 0; the opponent places seven markers of another color at 2. The first player tries to get his or her markers exactly on 2; while the second player tries to get his or her markers exactly on 0. If a player lands on the opponent's marker, that marker is sent back to the beginning location. The first player to win three markers wins the game.

4. Each player starts at 0 with seven markers of his or her own color. On a turn, a player can move forward on one of the lines, or move an opponent backwards on one of the lines, or save the card for a later turn in the game. (A player can only have one "saved" card in possession at a given time.) The goal is to win three markers by getting to 2 exactly.

5. Playing with common or individual markers, a player can, on each turn, move markers on all possible lines. For example, if the card is 1/4, a player can move 1/4 on the fourths lines and 2/8 on the eighths line.

(Note: The numberlines mat games are related to those used in Fraction Bars written by Albert B. Bennett and Patricia S. Davidson, Fort Collins, CO: Scott Resources Company, 1973.)

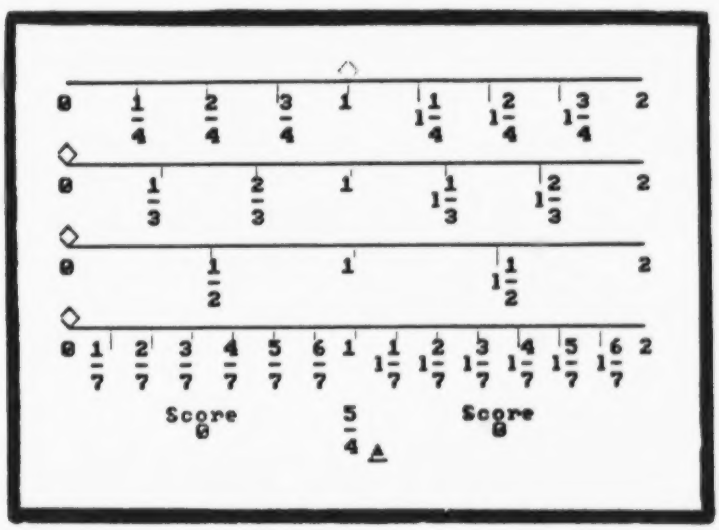
Stage 3 - Numberlines Computer Game

This is a two-player computerized version of just one of the Numberlines Mat Games as it is in an early stage of development. The program randomly chooses four types of numberlines (halves through tenths) and places them on the screen. Each numberline spans from 0 to 2 so that equivalent fractions line up vertically. This same set of numberlines remain on the screen for an entire game, but randomly changes for each new game.

At the beginning of a game, a diamond-shaped marker is placed on the 0 of each numberline. The program also randomly chooses the fraction for each player's turn. As in the mat games, players alternate moving the markers along the numberlines in the direction of 2. The arrow keys are used to move the markers at the particular intervals on the numberlines. For example, for the fraction 4/6, a player needs to push the arrow key four times to move 4 sixths on the sixths numberline or two times to move 2 thirds on the thirds line. A player can move markers to test out turns before pushing the asterick key to declare the final move.

If a fraction is generated that cannot be played on the present numberlines, the player can choose any fraction that is less than this random fraction, and one that is also able to be played on the given numberlines. For example, if 4/5 is the random fraction generated and the numberlines are halves, fourths, sixths, and sevenths, then the player could move an amount like 2/3 or 4/7.

When a player has arrived exactly at 2, a point is scored for that player and the marker goes back to 0. A player wins when he or she has accumulated a preset number of points. At the beginning of a game, the players can determine the number of points that constitute a game.



Stage 3 - Mixed Number Measurement Activities
Worksheet 3.3 and 3.4

Students use the rulers they made to measure lengths whose value will turn out to be non-integral lengths greater than one unit (mixed numbers). On Worksheet 3.3, the idea of equivalence is stressed since the given lengths can be expressed with several equivalent mixed numbers. On Worksheet 3.4, the idea of betweenness is stressed through the use of a sequence of rulers with successively finer subdivisions so as to define the length in question with greater and greater precision.

Materials:

Lengths of cardboard or other heavy paper
Five lengths of cardboard identified as A, B, C, D, and E (use with 3.3)

Cardboard A - 28 in. long
Cardboard B - 35 in. long
Cardboard C - 26 2/3 in. long
Cardboard D - 17 1/2 in. long
Cardboard E - 45 in. long

Five lengths of cardboard identified as F, G, H, I, and J (use with 3.4)

Cardboard F - 23 1/8 in. long
Cardboard G - 31 7/8 in. long
Cardboard H - 17 1/7 in. long
Cardboard I - 24 4/9 in. long
Cardboard J - 17 7/9 in. long

Worksheets 3.3 and 3.4 (1 or 2 copies per student)
Pencils
Undivided measuring strips
Fractional measuring strips - halves, thirds, fourths, fifths, sixths, eighths, and tenths

Stage 3 - Worksheet 3.3

Students choose one of the cardboard lengths (A, B, C, D, or E) to measure and identify it as Cardboard ___ in the appropriate place on Worksheet 3.3. They first try measuring with the wholes ruler which does not work for any of the cardboard lengths. Then they try consecutively the halves ruler, thirds ruler, fourths ruler, fifths ruler, sixths ruler, eighths ruler, and tenths ruler. Each time the students record whether or not the ruler works and if so, what the length of the cardboard would be in terms of the subdivisions on the particular ruler.

Each student measures two or four of the cardboard lengths and compares his or her measurements with other students in the group.

The focus here is on equivalence; for example, for Cardboard B, the length will be recorded as 3 1/2, 3 2/4, 3 3/6, 3 4/8, 3 5/10 in terms of the measuring ruler (10 in. strip) as the unit of measure. The values for the other Cardboard lengths are:

Cardboard A: 2 4/5 or 2 8/10
Cardboard C: 2 2/3 or 2 4/6
Cardboard D: 1 3/4 or 1 6/8
Cardboard E: 4 1/2, 4 2/4, 4 4/8, or 4 5/10

Worksheet 3.4

Students choose one of the cardboard lengths (F, G, H, I, or J) to measure and identify it as Cardboard __ in the appropriate place on Worksheet 3.4. These lengths should not have exact measurements in terms of the mixed number values on the worksheet. The students first try the wholes ruler and list the number of wholes that are "just too short" and the number of wholes that are "just too long."

The students then use the halves ruler and note how many halves, using mixed number notation, are "just too short" and "just too long."

Similarly, the students measure with the thirds ruler, fourths ruler, fifths ruler, sixths ruler, eighths ruler, and tenths ruler and finish the chart. The cardboard lengths have been chosen so that measurements are not exact. Successively finer subdivisions are needed to define the length with greater and greater precision. The students should record the closest measurement they were able to obtain to the length of each cardboard strip.

Each student should measure two or four of the cardboard lengths and should compare his or her measurements with the other students in the group.

The focus of this activity is on the expression of the measurement of a length as a mixed number value between __ and __, the first value being "just too short" and the second number being "just too long."

Worksheet 3.3

Use your measuring strips (rulers) to maasure the lengths of two pieces of cardboard. Record which pieces you are measuring (A, B, C, D, or E). Fill in the recording sheet with your results, using mixed notation.

1. Cardboard _____

Type of ruler	Does the ruler work?	If it works, what is the length?
Wholes ruler		
Halves ruler		
Thirds ruler		
Fourths ruler		
Fifths ruler		
Sixths ruler		
Eighths ruler		
Tenths ruler		

2. Cardboard_____

Type of ruler	Does the ruler work?	If it works, what is the length?
Wholes ruler		
Halves ruler		
Thirds ruler		
Fourths ruler		
Fifths ruler		
Sixths ruler		
Eighths ruler		
Tenths ruler		

Worksheet 3.4

Use your measuring strips (rulers) to measure the lengths of two pieces of cardboard. Record which pieces you are measuring (F, G, H, I, or J). Using mixed notation, fill in the chart for the values "just too short" and "just too long" for each type of ruler used. Record the closest measurement you are able to obtain to the length of each of the cardboard pieces.

1. Cardboard _____

Type of ruler	Just too short	Just too long
Wholes ruler		
Halves ruler		
Thirds ruler		
Fourths ruler		
Sixths ruler		
Eighths ruler		
Tenths ruler		

Closest Measurement Obtained: _____

2. Cardboard _____

Type of ruler	Just too short	Just too long
Wholes ruler		
Halves ruler		
Thirds ruler		
Fourths ruler		
Sixths ruler		
Eighths ruler		
Tenths ruler		

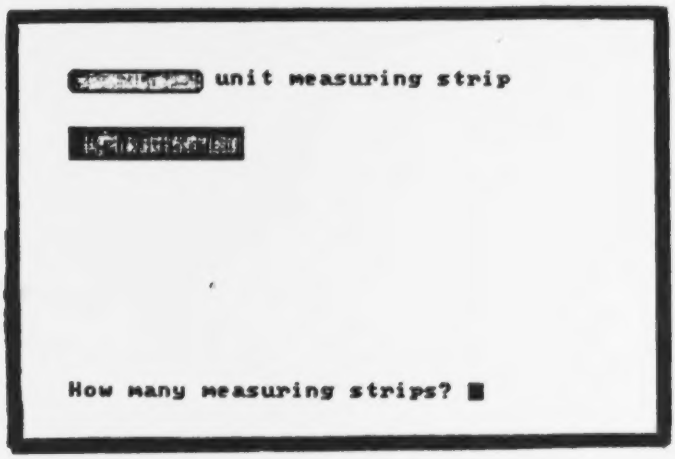
Closest Measurement Obtained: _____

Stage 3 - Mixed Number Measurement Software

This activity involves using a microcomputer-based game which is logically, and, to some extent, perceptually equivalent to Worksheets 2.1 - 2.3. The computer permits extension of those earlier activities to fractions not easily created by paper-folding. This game is also an extension of Worksheet 3.4 where all denominators are used through tenths and students are able to input more and more precise mixed numbered values for measurements of lengths shown on the screen.

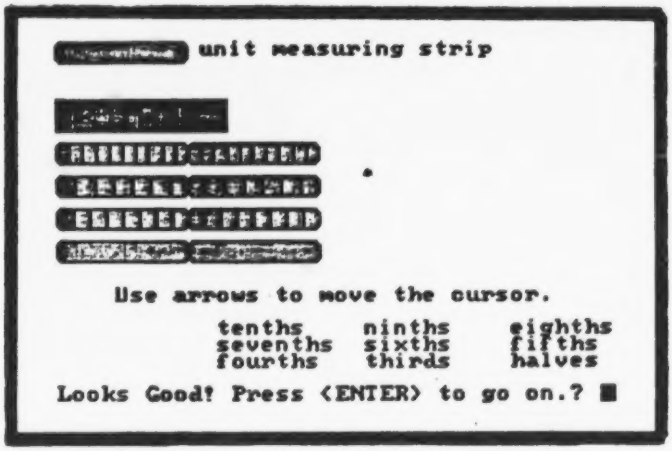
The mixed number measurement software was designed for the pilot study as an exploratory environment that would allow students to measure random lengths using various types of measuring strips. The number of choices of measuring strips is more extensive than those available in the prior experiences with folded measuring strips. In the current version of the game, the units can be subdivided into halves, thirds, fourths, fifths, sixths, sevenths, eighths, ninths, and tenths.

The unit length the players use is displayed on the introductory screens. Then a bar of a random length is drawn on the screen. The players first determine the appropriate number of whole units needed to measure the bar.

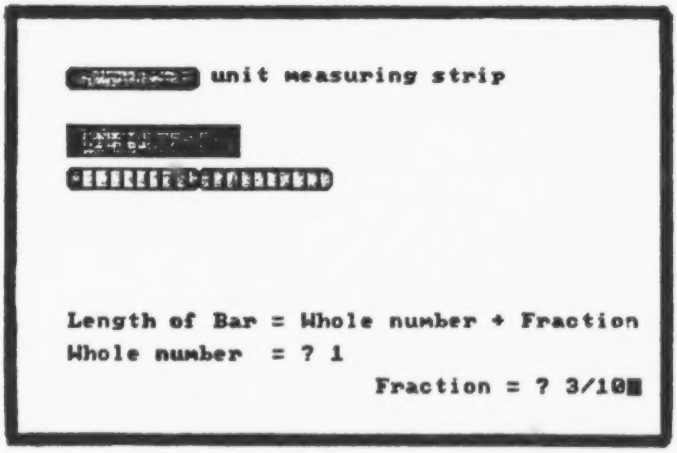


Once that number of measuring strips is correct, then a menu of fractional subdivisions is displayed. Players select which type of measuring strip they want to use. The type of measuring strip selected is displayed on the screen with the appropriate subdivisions. Additional choices can be made. Subsequent selections are displayed on the screen with the most recent try directly under the bar as the other tries scroll downward. There can be up to four tries on the screen at a time.

When a match occurs, an appropriate message appears. Players are then instructed to press RETURN which sets up the next screen. Here, the bar and measuring strips with correct subdivisions are displayed. The player then types in the mixed number notation which expresses the correct length. Any equivalent fraction representation will be accepted. (e.g. if the bar is 2 3/4 units long, all of the following will be interpreted as correct input - 2 3/4, 2 6/8, 1 7/4, 1 14/8, 11/4, or 22/8.) If an incorrect fraction is typed in, a bar of that inputted length will appear on the screen, thus illustrating its inappropriateness.


unit measuring strip
Use arrows to move the cursor.
tenths ninths eighths
sevenths sixths fifths
fourths thirds halves
Looks Good! Press <ENTER> to go on.?

After correctly representing the mixed number or fraction, the player is asked if he or she wants to play again. If the input is yes, then another bar is displayed on the screen.



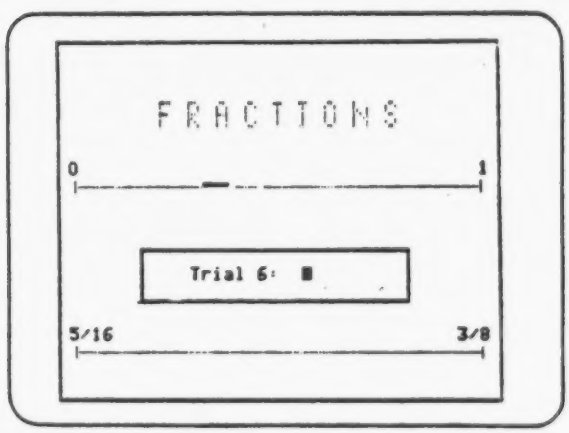
Stage 4 Order Relationship of Fractions and the Notion of Betweenness

Stage 4 - Number Quest

The activities in this stage were included to give students opportunities to work more with the ordering of fractions.

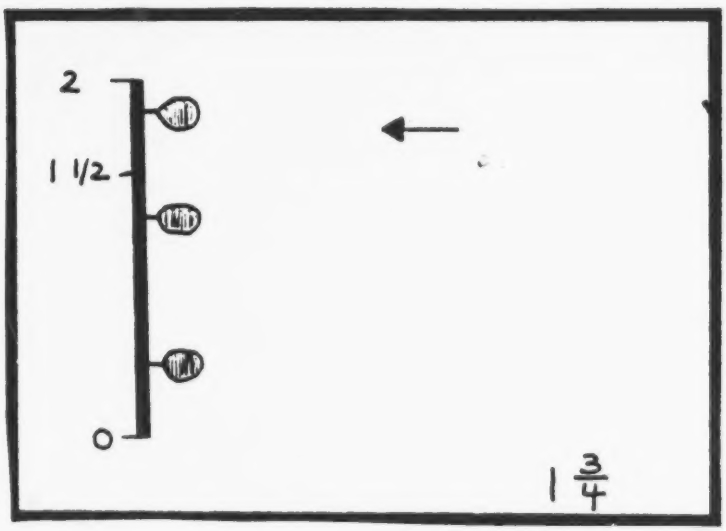
Number Quest (published by Sunburst) is a set of five programs, each of which challenges the user to find a number or set of numbers that the program chooses at random. The program pertinent to Stage 4, entitled FRACTIONS, asks the user to find a randomly chosen fraction between 0 and 1. Utilizing the Change Option Menu, we limited the fractions being chosen to those with denominators less than or equal to 8.

Students are presented with a number line labeled 0 to 1 and are asked to type in a fraction between 0 and 1. After one trial, the original 0 to 1 number line is pictured on the screen with the narrowed guessing area highlighted on the numberline. Also a second numberline is presented below which magnifies the narrowed guessing area and end points. Students continue guessing until they find the particular fraction which the program had chosen at random within the limits set by the Change Option Menu. The program records the number of trials needed. If the fraction typed in is not within the highlighted area or is an equivalent to one of the end points, the program beeps and won't take the answer.



Stage 4 - Darts

Darts (published by Apple Computer Inc.) is a game where a numberline is presented vertically on the screen with 3 randomly placed balloons on it. The end points and one randomly placed point are labeled to indicate unit size. The object of the game is to pop the balloons by typing in a number (fraction, integer, or mixed number) as close as possible to the location of the balloons. Once a number has been inputted, a dart moves across the screen from the right side at the location of the number typed in. If it is very close or a direct hit, the balloon will pop and the number will appear on the numberline. If the dart misses the balloon, the location of the guess will still be indicated on the numberline, thus helping the user refine the next guess. A game is over when all balloons have been popped. The program records the number of trials.



PILOT STUDIES

The major goal of the two pilot studies was to do enough teaching of the activities to see if they were engaging and understandable to students and to obtain information for the revision of the materials.

December Pilot Study

In December 1984, research assistant, Anne Dickenson, conducted a pilot study with two groups of students (grades 5 and 7). At that time the Stage 2 and 3 activities were ready to be taught. The pilot study was intended to answer two basic questions:

1. Do the activities interest the students and engage their attention?

2. Are the activities substantively appropriate to the ages/grades for which they were designed?

Subjects

Ten children in grades 5 and 7 from the Watertown, Massachusetts Public Schools participated in the pilot study. The children were randomly chosen from volunteers of classes of average ability.

Procedures

Two groups of five students each worked on the activities for four periods, each lasting between forty and sixty minutes.

The research assistant took notes during the sessions including students' comments and their involvement with the materials.

Conclusions

Both questions were clearly answered in the affirmative. The students at both grade levels were interested in and engaged by the activities. They liked the playfulness of the hands-on activities and the interactiveness of the computer activities. The activities seemed substantively appropriate because students often gained insights from the activities. Based on a review and analysis of the students' responses, the working group refined and revised the Stages 2 and 3 activities and developed activities to provide a smoother transition between Stages 2 and 3.

Next, the group developed the activities for Stages 1 and 4. The complete set of activities, as described in the previous section of this technical report, were piloted in May 1985.

May Pilot Study

When the activities for stages 1 and 4 had been developed and those for stages 2 and 3 had been revised to make a better transition between the stages, the working group conducted a second pilot study. In addition to students' affective responses to the new activities, we wanted to discover which activities needed to be further refined or put in a different sequence and to use the activities to begin to learn about students' strategies for understanding equivalence, order and betweenness.

Subjects

Five students in grade 5 and five students in grade 7 from the Newton, Massachusetts Public Schools worked with research assistant Cornelia Tierney, and five students in grade 5 and five students in grade 7 from the Watertown, Massachusetts Public Schools worked with research assistant, Anne Dickenson.

The fifth grade students were volunteers drawn from regular classes at the Lowell School in Watertown and the Countryside School in Newton. The seventh grade students were drawn from a middle track class at the West Marshall School in Watertown and from a lower track class at Day Junior High School in Newton.

Procedures

Sessions took place during May on eight to eleven days for 40 to 60 minutes each. The Countryside group met after school in their math classroom. The other groups met during their regular math time in rooms away from the regular classrooms.

To aid in the analysis of student responses, the research assistants wrote detailed notes immediately after each session (See Appendix A). In addition, members of the ETC Fractions Group acted as observers at the teaching sessions, and the sessions were recorded on audio-tape to capture the interactions between teacher and students and among students. Each observer was asked to address a list of questions (See Guidelines for Observers, Appendix C). These included:

What revisions and/or changes would you suggest to improve each lesson?

What insights and strategies did the learners use?

The observations of the research assistants and other group members were the basis of discussion and conclusions about further revisions in the teaching activities, which are described in the next section.

CONCLUSIONS: PROPOSED CHANGES AND REVISION OF STAGES 1-4 ACTIVITIES

The following conclusions are the result of discussion and analysis by members of the Fractions Group based on their observations. Detailed changes and revisions are listed for each stage. Also included is the rationale leading to the decisions and the remaining unresolved issues. Where the rationale is not explicitly stated, the reason for revision was the students' confusions about what was expected.

Stage 1 Simple Length Measurement: Integer Number of Units

- o Keep the first activity and Worksheet 1.
  - students easily grasped principles of measurement using this activity
- o Eliminate the height measuring activity because
  - the activity took too long and was boring,
  - it seemed too young and was embarrassing to the students,
  - the sameness of the two worksheets (1.1 and 1.2) caused confusion,
  - the activity encouraged poor iteration techniques, and
  - it did not add anything new.
- o Keep the activity where students create measuring tapes using adding machine paper and tie the use of these measuring tapes to the Stage 3 activities so as to integrate the stages and provide a more accurate way to measure the lengths in Stage 3.
- o Still resolve several issues:
  - whether a tape is needed for each unit of measure,
  - whether tapes should be made by the students just for the experience, but then whether accurately made tapes should be provided for the actual measurement activities.

Stage 2 Measuring Strip Activities

- o Keep the folding fraction strip activities, but
  - mark the fold lines for thirds and sixths on the strips to help alleviate the frustration and confusion in folding for those who find this difficult,
  - be more explicit about labeling the fraction lengths at the interval marks rather than within the spaces; model one or two examples before students do their own,
  - start out the labeling of each strip with marking the 0 and whole first so that the students focus on the unit length, and
  - suggest folding the sixths strip in half to check on the accuracy that the fold for 3/6 comes exactly at 1/2.

- o Have accurate pre-made measuring strips available for those students whose folded ones are inaccurate.

- o Keep Worksheets 2.1 - 2.4, but
  - explain the directions orally as well as in writing for the worksheets to focus students' attention on instructions,
  - do in the order 2.4, 2.1, 2.2, and 2.3 since Worksheet 2.4 encourages the use of strips and helps the students to think of fractions as lengths,
  - change the order of the lines on Worksheet 2.1 to be in decending order (halves, thirds, fourths, fifths, sixths, and eighths) to make it more consistant with the number lines in Stage 3,
  - foster the notion when using Worksheet 2.1 of betweenness and equivalency by posing questions like:
    Name a fraction equivalent to 1/3.
    Name a fraction between 1/3 and 2/3.
    Which is larger 1/6 or 1/8?

- o Introduce the computer software, Number Quest, limiting the denominators to eighths and making available to the students a numberlines mat similar to Worksheet 2.1, but including fifths and sevenths since these fractions are used in Number Quest.

- o Consider the possibility of having students who play Number Quest with the a numberlines mat use the optional Projection Activity to verify the fifths and sevenths subdivisions.

Stage 2 Island Hopping Games and Island Hopping Computer Games

- o Keep the paper board games and the computer games.
- o Complete and refine the software.
- o Modify the rules so that drawing a 0/n card means that the player can choose another card from the deck (or use it as a wild card which allows the students to choose any fraction) to make the game move faster and to open more game options.
- o Make sure that there is ample time to play several versions of the games, instituting both the paper games and computer games as ongoing activities in the classroom throughout the year,
- o Encourage the development of new rules by the students and teachers.

Stage 3 Part - Whole Relationships Activities

- o Do some preliminary activities where the students can realize the effect on fractional parts when the unit changes, for example,
  - fold halves of different unit lengths and see that 1/2 is not always the same length; one must ask, "1/2 of what?",
  - use Cuisenaire rods to show parts of wholes for diffferent-sized units, and
  - show, for example, that the red Cuisenaire rod is 1/2 of purple, 1/3 of dark green, 1/5 of orange; a given length can represent different fractional parts when the unit changes.
- o Change the format and content of Worksheets 3.1 and 3.2,
  - have the students extend the fractional drawing on the first line rather than including a second line on which to draw,
  - change the order of the problems on Worksheet 3.1; in the present form, the top half of the problems logically follows the bottom half of the problems,
  - make a worksheet that goes from parts to wholes focusing on the same fraction, for example,
    if this is 1/5 _____, draw the whole
    if this is 1/5 _____, draw the whole, etc., and
  - make more problems like the last one on Worksheet 3.2 as an option for those students who work rapidly and have extra time.

Stage 3 Numberlines Mat Activities and Games

- o Incorporate into the instructions a number of options which were successful in the pilot study:
  - Let the students keep the markers that they have won, at least until the end of class, making the game more interesting.
  - As an extension of all the numberlines activities and games, include fifths, sevenths, and ninths on the mat and in the deck of cards.
  - Allow students to split up the fractional lengths so as to move on more than one line for a given turn.
  - Encourage the students to use the Numberlines Mat (halves through tenths) when playing with the Number Quest, Mixed Number Measurement, and Darts software.
  - Use the Numberlines Mats to find a fraction between two given fractions.
- o Add the following materials for ongoing use:
  - Make an enlarged Numberlines Mat (halves through tenths) available as a visual bulletin board in the classroom.
  - Make 8 1/2 in. X 11 in. Numberlines Mats available for students to use for reference on a regular basis.

Stage 3 Numberlines Computer Games

- o Eliminate the Numberlines Software in its present version of moving horizontally along the lines since it is too slow and very limiting in comparison to the mat games.
- o Revise the Numberlines Software to illustrate the order and betweenness aspects of fractions by having the computer be able to place underneath each other or merge on top of each other, two numberlines with different subdivisions. The focus here is on the vertical "lining up" properties of the array of numberlines and the inclusion of one set of fractions within another set of fractions, such as the fourths subdivisions being included in eighths subdivisions.

Stage 3 Mixed Number Measurement Activities

o Keep Worksheet 3.3, but,

- change the wording "Does the ruler work?", to "Does the ruler fit?", as suggested by some fifth grade students,
- use measuring tapes accurately drawn on adding machine paper especially with the students who have problems iterating correctly, and
- make explicit the equivalence of certain measurements.

o Eliminate the cardboard strips in Worksheet 3.4; instead measure real-life objects, especially the same ones measured in Stage 1 prior to fractions being introduced. Use the same format of Worksheet 3.4 to determine measurements as being between ___ and ____.

Stage 3 Mixed Number Measurement Software

o Revise the following aspects of the software:

- Fix the bug in the program that does not accept equivalent fractions,
- Add a feature to the program that keeps track of how many guesses are used to determine the length of each strip.
- Redo the way to input mixed numbers to be more like conventional notation.

o Design additional components which:

- Encourage students to use equivalent names for lengths.
- Help students to use their knowledge of equivalence in playing the game; for example to realize if they have tried eighths, they do not need to try fourths since the fourths are included in the eighths.

Stage 4 Order Relationships of Fractions and the Notion of Betweenness

- o Strengthen the transitions from Stages 3 to Stage 4.
- o Include additional activities to provide practice with ordering and betweenness of fractions:
  - Encourage the students to use the Numberlines Mats when playing both Darts and Number Quest,
  - Play Number Quest earlier (possibly in Stage 2) with denominators up to 8 and in Stage 4 with denominators through 16,
  - Work out activities which help the student develop a variety of strategies to find a fraction between two fractions and to order a set of fractions:
    - considering the case with like denominators
    - considering the case with like numerators
    - considering the case with unlike numerators and denominators.
  - Expand the Island Hopping Computer Game to include work with betweenness; create a playing option where one fraction will be randomly generated by the computer, and the player will have to enter a smaller fraction or a larger fraction to determine the length of the move; create a playing option where two fractions will be randomly generated by the computer and the player will have to enter a fraction between these two fractions to determine the length of the move.

The final sessions of the May pilot study found the students trying different ways of determining what fractions to choose as they toyed with some fairly complicated issues of order and betweenness as posed in the pieces of software in Stage 4. It is these very strategies that the ETC Fractions Group wants to capture and define in their next research efforts during 1985-1986.

PLANS AND TIMELINE FOR ACADEMIC YEAR 1985 - 1986

The timeline and next steps for the ETC Fractions Group are summarized in the table below and are then followed by an elaboration on each aspect of the future research plan:

15 September 1985	Make revisions on Stages 1-4 activities. Develop Numberlines Environment Software. Design pre-test and post-test experiences. Devise clinical interviewing procedures. Organize four research groups in the schools.
1 December 1985	Conduct student interviews with eight of the 20 students from the May 1985 pilot study as a pilot study of the clinical interviewing process.
15 January 1986	Administer pre-test experiences to the groups involved in the research study. Start teaching Stages 1-3 activities in small groups.
1 March 1986	Submit progress report to ETC.
15 March 1986	Start clinical interviews with individuals or pairs of students and observe them on Stage 4 software.
15 April 1986	Administer post-test experiences to the students involved in the research study.
15 May 1986	Begin analysis of data and report writing.
1 September 1986	Submit draft of technical report to ETC.
30 September 1986	Submit technical report to NIE.

Revisions

In September and October of 1985, the ETC Fractions Group will revise the Stages 1 - 4 activities and materials according to the specifications outlined in the pilot study Conclusions. These suggestions for refinements in the learning experience come directly from the student interractions with the materials.

Software Development

The Numberlines Computer Games will be replaced with a Numberlines Computer Environment which will allow students to maneuver more than one number line through vertical placements, vertical lining up, and vertical mergings so that students can explore issues of order and betweenness. The computer will be able to generate accurate number lines and then place them underneath each other or on top of each other to show comparisons for

fractions of different denominators. The focus will be on the vertical "lining up" properties of an array of number lines, on the density property of the set of rational numbers, and on the inclusion of one set of fractions within another set of fractions, such as the fourths subdivisions being included in eighths subdivisions. These ideas will strengthen the transition from Stage 3 to Stage 4.

Development of Assessment Instruments

From the beginning of the project, members of the Fractions Group have talked about the need for pre-test and post-test experiences to help define more precisely what the students actually learn from the Stages 1 - 4 activities. The tasks will need to include a merging of two viewpoints, namely the types of fractions to be used and the types of situations to be posed. The types of fractions to be included are:

Fractions < 1

unit fractions	e.g. 1/2, 1/3, 1/4
non-unit fractions, same numerators	e.g. 3/5, 3/4, 3/7
non-unit fractions, same denominators	e.g. 3/5, 4/5, 2/5
fractions with different numerators and denominators	e.g. 2/3, 3/4, 5/6

Fractions > 1

fractions with same numerators	e.g. 8/5, 8/7, 8/3
fractions with same denominators	e.g. 7/4, 9/4, 10/4
fractions with different numerators and denominators	e.g. 7/4, 8/3, 9/5

The tasks to be considered are:

1. Given a fraction, name three that are equivalent.

2. Given a fraction, name one that is smaller.

3. Given a fraction, name one that is larger.

4. Given two fractions, determine which is larger (smaller).

5. Given a set of fractions (five or more), put them in order from smallest to largest (largest to smallest).

6. Given two fractions, name a fraction that is in between these two (without and with restrictions on the denominators).

Since the major thrust of the research study will be on capturing and defining the various strategies that the students use to deal with issues of order and betweenness, the ETC Fractions Group will devise a careful clinical interviewing process to be used at Stage 4 while the students are working with the pieces of software. The clinical interviewing process to be used will be based on both the successful experiences of past researchers and the specific needs of the issues being addressed in this project.

Some strategies for ordering fractions have already emerged from observations during the pilot study. They can be summarized as follows:

a. for fractions with the same numerator, compare the denominators (inverse relationship).

b. for fractions with the same denominators, compare the numerators (direct relationship).

c. for fractions with unlike numerators and denominators,

   i. use the array of number lines or number strips and make vertical comparisons.

   ii. use a frame of reference such as 1/2 and the transitive property; for example, 4/9 < 5/8 because 4/9 < 1/2 and 5/8 > 1/2.

   iii. use a residual strategy;
   for example 5/6 < 7/8 since the pieces that make up the whole can be compared easily. Since 1/6 > 1/8, then 5/6 < 7/8.

   iv. use equivalent fractions with a common denominator and compare the numerators of the equivalent fractions (direct relationship).

d. for fractions greater than one, convert both of them to mixed numbers and

   i. if the whole numbers are not alike, compare the whole number part and ignore the fractional parts.

   ii. if the whole numbers are the same, compare the fractional parts using the above strategies for fractions less than one whole.

No doubt other techniques will emerge and greater insights will be gained into how students think about ordering fractions. In our research, students will not be taught techniques of maneuvering symbols such as cross multiplying or converting to decimals.

The issue of how students place a fraction between two other fractions is even more open for exploration since very little research has been done on betweenness issues. When restrictions are made on the size of the denominator that may be used, students are faced with a challenge which the common denominator techniques cannot solve. However, some students may know such techniques from past classroom experiences, and if so their techniques will be explored.

Research Study

The general research questions to be addressed are as follows:

1) What correct and incorrect strategies do students use in their attempts to order fractions?

2) What correct and incorrect strategies do students use to find a fraction between two fractions (both when any denominator may be used and when the size of the denominator is limited)?

3) Do children who have worked through the experimental curriculum demonstrate the ability to order fractions successfully?

4) In what ways do the experimental activities and material help students to understand fractions more clearly as numbers between numbers?

In developing the clinical interviewing process, the ETC Fractions Group will need to more carefully delineate each of these questions into a series of subquestions. After the clinical interviewing process is developed, it will be piloted in December with eight of the 20 students from the May 1985 pilot study. These students will need a quick review of the Stages 1 - 3 activities in which they participated and then will continue into Stage 4 with close observation and documentation.

The ETC Fractions Group will contact the schools involved in the collaboration in November to set up four new research groups, again from grades five and seven. Each group will consist of five to six students so that they can be observed closely. Then in January, the pre-test experiences will be administered to the groups, and the post-test experiences will be administered in April following the teaching sequence and clinical interviews. The Stages 1 - 3 activities will be taught to the entire group; whereas the clinical interviews in Stage 4 will be conducted with individuals or pairs of students.

Analysis and Documentation

The analysis of all of the data from the pre-test and post-test experiences, the teaching of Stages 1 - 3, the clinical interviews in Stage 4, will begin in April. A detailed Technical Report will be written by the ETC Fractions Group during the summer and will be submitted in September 1986.

Staffing

In September 1985, the ETC Fractions Group hired an additional 1/3 time research assistant. It also added a teacher from the Cambridge Public Schools so that all four school systems in the collaborative are now involved in the research study.

APPENDIX A: Anecdotal Notes of Pilot Study - Part 1
Newton, Massachusetts Public Schools
Cornelia Tierney, research assistant

(It is suggested that these anecdotal comments be read after and/or along with the Descriptions of the Stages 1 - 4 Activities section of this report.)

Stage 1: Simple Length Measurement: Integer Number of Units
Worksheet 1.1

Once the students had measured the width of an object with an unsharpened pencil as the unit of length--the object measured in the seventh grade was a table and in the fifth grade a windowsill--I gave each student an unsharpened pencil, an orange Cuisenaire rod, a straw, and a paper strip to use for making estimates of how many of each would be required to measure the same object. Most of the students made comparisons of the units in trying to figure quite exactly how many of each of the units would be required.

Before the students checked their estimates by actual measurement, I asked them to tell the group their answers and how they had figured them. The students who realized that because a rod is about half the length of a pencil twice as many would be needed were able to explain their reasoning clearly enough to convince other students who had not thought of that strategy. Student D, a fifth grader explained his strategy this way, "I compared it. This [pencil] is almost two [rods]. I figured double six, then add half."

Our discussion seemed to help the thinking both of the speakers and of the listeners. R's explanation to the seventh grade group convinced Q to change her answer; Q had initially guessed the same number of rods as pencils. A fifth grader, C, when asked to explain her answer, caught her own mistake of halving the number of pencils to get the number of rods.

Only B, a fifth grader, never compared the units to make estimates. Instead, she estimated spatially by holding the units up in front of her while standing at a distance looking at the windowsill she was trying to measure. When I asked if that worked, she said it didn't; but later when she tried it again she said, "It works for me."

A number of fifth graders who had willingly reported measurements in a range "between __ and __" tried to be more precise in their estimates by using fractions. For example, F guessed 5 3/4 straws for the windowsill which was a little over 6 pencil lengths. I asked if they thought guesses with fractions were more likely to be more correct than those with just whole numbers. Several said they thought so, but one of the girls, C, said, "No, a number is just a number no matter how you put it. It's just as close or far from the right answer."

The students became very involved in this activity. Measuring with the pencil first and then working independently to figure estimates worked well as a first lesson.

Stage 1: Worksheet 1.2

The activity involving the measuring of heights did not go as well. It embarrassed some fifth graders who thought it was babyish. They asked me to take the measuring tapes off the wall before the sixth grade saw them. A seventh grader, P, commented when we were almost finished, "This is boring."

An interesting confusion occurred when the students used measuring tapes, one which Brenda Roberts, the seventh grade teacher, had mentioned had also been a problem for these students in the fraction work in class. Some students did not know whether to count the marks which they wrote as they iterated the units or whether to count the spaces. For example, when P was measuring the height of another boy against the blackboard on which rod lengths were marked, he first counted marks including the one at the bottom of the board and got a total of 13. When I suggested he count again, he placed the palm of his hand in each interval and counted 12 which he decided was correct.

Stage 2: Measuring Strips Activities

The same confusion of endpoints vs. spaces appeared again in the folding and labeling of strips. One fifth grade student, B, labeled some spaces and some folds and was not phased when she got to 9/8 for the whole instead of 8/8. When P (mentioned above) was labeling a measuring strip, he also at first wrote the fractions within the spaces, most likely influenced by the method he had used to count up the number of units in the height of the boy.

Some students had great difficulty folding the strips into equal pieces, especially into thirds, sixths, and eighths. D kept folding and unfolding the thirds until he was satisfied that they were accurate. He seemed to enjoy the challenge and did not mind that other students were working ahead on the worksheets. In contrast, B in fifth grade and R in seventh hurried through all the paper folding and only made some second tries because I asked them to do so. They did not seem to have any idea why we were doing this or what the end result ought to look like. Perhaps after the folding of halves and fourths, it would have been better to give these two students the pre-prepared strips for thirds, sixths, and eighths.

After F said she did not need to cut sections to use for making marks, all the fifth graders attempted to label creases without using the pieces as guides. When B became confused about folding and labeling, I suggested that she cut pieces to use. When she did this, however, the pieces she cut were more like trapezoids than rectangles. One edge was too long and the opposite too short for marking off equal intervals. In seventh grade, as soon as I saw that R and T were cutting inaccurately, I suggested they just fold and label the creases. Using this shortcut, however, meant that they missed out on an experience which might have helped clear up the confusion about whether to label marks or spaces.

Stage 2: Worksheets 2.1 - 2.4

The confusion about marks or spaces occurred once more on the worksheets. P explained to T who labeled eighths as ninths on Worksheet 2.1 that he should count spaces instead of endpoints. After some discussion, both P and T labeled their worksheets correctly at the interval marks.

The students least liked Worksheet 2.2 which appeared to them like a repeat of 2.1 except that only points marked with letters were to be labeled. When I asked R why he wasn't getting started with it, he said, "I know what to do, but I don't know what those letters are supposed to mean."

For him as well as most of the others, Worksheet 2.4 was the favorite. In this activity, the students used felt markers to color in the intervals and label the endpoints. Worksheet 2.4 is the first of the worksheets which actually requires using the marked strips. Because Worksheet 2.3 has marks only at unit fraction lengths from an endpoint, some students were able to do it by estimating how many of the smaller intervals are in the larger interval. (For example, in the first problem where the mark is at 3/4, three iterations of the interval to the right of the mark can fit into the interval to the left.)

To acquaint the students with the use and the need for the strips, Worksheet 2.4 could be put first and have more problems on it. Certain problems could be put next to one another to point out differences between fractions, for example 3/4 and 5/6, which many students think are the same. Because Worksheet 2.2 is so much like Worksheet 2.1, it could be omitted or be made optional.

To occupy students who finished early, I asked them to sort fraction cards in order of size. A seventh grader put 2/3, 3/4, 4/5, and 5/6 together as the same size. This activity led to a discussion of these fractions as being close to one whole and of the missing parts, 1/3, 1/4, 1/5, and 1/6. When I asked S if she would rather give away 1/5 or 1/6 of a million dollars, she immediately responded, "1/6 because it's smaller." But she couldn't explain why, until a visiting observer used one of the rods to draw two equal rectangles and asked her to divide one into fifths and one into sixths. She stayed after class to figure out for herself a comparison of the two fractions, 1/5 and 1/6. After dividing the rectangles into five and six pieces respectively, she had a real "aha" reaction when she saw that if there are more parts of the same whole, each is smaller. Even though this model was almost exactly like the folded strips, she had not made these observations earlier.

Stage 2: Island Hopping Games

The part that I found most interesting was observing how the students picked fractions when they had a choice as in version 3. Some simply knew which fractions to pick by looking at the numbers, while others narrowed down

the choices by first discarding zero (which meant no move) and wholes (which were too big for most moves) and then using the strips to determine which of the remaining ones would fit best. Two students seemed to pick randomly. The game requiring the most strategy was version 5, a solitaire game in which the students planned in what order to use five fractions which were dealt to them. C, D and F in the fifth grade and P and S in the seventh grade were able to do this quite successfully. Although it wasn't a favorite activity, it kept them busy for a while and gave me a chance to work with other students.

R in seventh grade worked very slowly. When in version 3 he was to pick a fraction, he looked at P's face for clues about which fraction to pick. R did not put the beginning of a strip down where the end of the previous one had been. He seemed to concentrate only on the fraction of the strip he was drawing without noticing where the beginning of the strip was.

Students were willing to play the group games, but none of them seemed particularly interested. For those who could draw the lines quickly, it was a bit boring in versions 1 and 2 to wait until all players had drawn their lines.

Stage 2: Island Hopping Computer Games

Because the other versions of the computer game were not ready, we played option 3 in which students could name any fraction including improper fractions. This brought up many questions about relative size of fractions and equivalent fractions, for example, which is larger 5/6 or 6/5? S explained that if the numerator is larger than the denominator, the fraction is larger than one whole, but when he needed a fraction a little larger than one whole he couldn't think of any. The computer program which is still in an infant stage of development had many bugs in it, but one turned out to be a real plus. The program refused the digit 1, so the students had to provide other names for 1/2, 1/4 etc.

As the first of the computer games, it proved to be a particular success with some students who had been unassertive up to that time. B in fifth grade made small moves, usually 2/4 or 3/4 and won against A who chose larger fractions to try to get to an island in one turn. In the seventh grade, Q suddenly asserted herself and told the others to be serious and hurry up to make their guesses. She played at first with a partner against two others, but she enjoyed the computer game more when she had a chance to play alone against one student while the other students played the pencil and paper version again.

Stage 3: Part - Whole Relationships Activities
Worksheets 3.1 and 3.2

The activities on Worksheets 3.1 and 3.2 separated the students of varied abilities more than any other activity. Students who had been able to pick cards without testing in the Island Hopping Game were able to pick rods by eye for these worksheets. They sped through all problems except the last

on Worksheet 3.2. Others needed to try one rod and then another. Q in seventh grade did not pick the rods in any consistent order. In fact if the first rod she tried was too short, the next one she chose might be even shorter.

Even for the quickest students, the last problem on Worksheet 3.2 which uses two different denominators (given 4/9, find 2/3) took a great deal of thinking. One of the four remaining fifth graders who sped through the other problems did this problem by finding one whole and then 2/3 of it; two others noted that 2/3 = 6/9 and added two more ninths to the 4/9 given. But the fourth, E, who had had no trouble with the rest of the problems became so upset trying this problem that he left the room for a while. The weakest fifth grader was absent. Two of the seventh graders, Q and R, had no strategies for doing this problem.

A whole page of problems with two different denominators would keep the faster students busy and provide a challenge to their thinking. In terms of format, all of the students I worked with merely extended the fractional drawing on the first line instead of using the second line; hence I feel that the second lines should be omitted on Worksheets 3.1 and 3.2.

Students suggested some changes in these worksheets. D in fifth grade was confused at first because 1/4 was the same physical size as 1/5 on Worksheet 3.1. He did not realize that we had used wholes of different sizes. He suggested making this clearer at the beginning of the worksheet. E did Worksheet 3.1 from bottom to top and found it very easy. Others who did it in the normal order did not notice when "one whole" was no longer asked for. Perhaps we should also reverse the order of the sections.

Stage 3: Numberlines Mat Activities and Games

In the seventh grade, the students read all the fractions off the numberlines mat in order of size while R wrote them on the blackboard where they stayed for the duration of the sessions. Later Q and T at different times suggested looking at that list to find fractions to guess in games. We also looked at the numberlines mat for largest and smallest fractions and for some equivalents. When I asked where on the mat were the largest numbers, Q pointed to 1/2. Other students told her that wasn't right but she wasn't convinced. Later that day I realized that what she meant was that 1/2 was the largest interval.

Some seventh graders also started to try the activity of putting two fingers on fractions on the board while their eyes were closed and then noting what fractions were in between, but they found it difficult and not much fun.

When A in fifth grade tried this, she touched 2/5 and 3/10. She immediately said that 2/5 equals 4/10 but that there is no fraction between 3/10 and 4/10. I asked what was between 3 and 4 and she said 3 and 1/2, but that the teacher wouldn't allow them to use fractions such as 3 and 1/2 tenths. She finally allowed herself to consider the possibility and then converted it to 7/20. She used this method of mixed number numerators later to find fractions between two numbers when she played Number Quest.

Although the seventh graders used the numberlines mat to find fractions between fractions for the later computer games, they never became really interested in the numberlines mat games. This was partly because in the original version, the game became boring when few cards gave fractions which were short enough to be used. This was corrected later by F in fifth grade who suggested that every penny which reached the end of a line should be replaced by one at the beginning. I also sensed that the seventh graders (at least all but J who was absent) have had little experience using strategies in games. They seemed to see this game mat as one more worksheet with right or wrong answers.

In contrast, the three strongest students in the fifth grade, who were also those with the best attendance, became very involved inventing and playing numberlines mat games. I brought the pennies to the fifth grade because they are heavier than the cardboard place markers we had initially been using in the seventh grade. F, the same student who had suggested that we put a penny on the beginning of a line to replace one taken off at the end, asked if they could keep the pennies they won. She wanted to call the game "Win a Buck," but I suggested she settle for "Win a Nickel."

Watching F and C play the numberlines mat games, I noticed that they had very different ways of determining where a piece was to be moved. F computed in her head and then put the piece where it was to go. C looked at a strip to see how long the fractional length of the move was and then estimated by eye how to move the piece that far. Once when she seemed unsure about her move from 3/8 to 1 3/8 and I pointed out to her the names of the points she had moved between, she was quite amazed that the numbers might give a clue as to how to move. It did not occur to either of the girls to count out the moves to be made.

The Numberlines Computer Games were not ready for this portion of the pilot study.

_Stage 3: Mixed Number Measurement Activities
Worksheets 3.3 and 3.4

In measuring long strips of cardboard with the unit strips, students seemed to concentrate only on the fractional part of the measurement, not on the wholes. Several of them were interested to notice that some lengths had several names, and those who already knew equivalents knew before trying that some strips would not work. Most used as many unit strips as needed and measured accurately, but one fifth grade boy, E, used only one strip and iterated inaccurately without marking the end of the strip where the next should begin. Hence he obtained different answers for the same measurement, but hardly seemed to care.

The students did Worksheet 3.4 with the same long strips of cardboard they used for Worksheet 3.3. This worked well except they needed a middle column for the measure which was "just right." If they had instead measured objects in the room, this would not have been a problem because it is unlikely any measures would have been "just right." Also measuring real-life objects would make the activity more relevant to students and would provide a natural link to the Stage 1 measurement activities.

Stage 3: Mixed Number Measurement Software

The fifth graders preferred to play this computer game as a solitaire. (Meanwhile other students played the numberlines mats games.) Because of a bug in the mixed number measurement program, denominators of equivalent fractions were not accepted. This was quite confusing and will need to be corrected for further use.

The seventh graders worked together with this computer activity for a full period. Again Q took charge. She explained the game to a student who came late, and she reminded others when it was their turn to guess. The students' suggestions for denominators seemed almost random. Some of their names for the fractions at the end of the game were accurate, but at times the students would suggest any answer, often an answer which had worked for a previous round. For example, the answer 13/10 from a previous problem was suggested although it had been established that the denominator was something other than tenths and that the answer was larger than two wholes. Students will need to play the game over a longer period of time in order to develop meaningful strategies.

After an input mistake led to a fraction with a numerator of 31, S became interested in testing the tolerance level of the program by entering what he called "ludicrous numbers." This included some correct answers such as 2 and 50/100 which the program didn't accept for 2 and 1/2. The software in its present state was effective but needs some further refinements.

Stage 4: Order Relationships of Fractions and the Notion of Betweenness
Darts and Number Quest

For the seventh graders, playing Darts was the high point of the sessions. It raised some good questions about number size. When students couldn't think of a number to guess, they looked at the numberlines mats or at the list of fractions still on the blackboard. Within each round, students took turns making guesses. This game provided a chance for using overall strategies--taking a broad view of the whole numberline--but few students did. They focused on small intervals instead of the whole picture. For example, instead of looking at the given endpoints and dividing the whole range in half to find a midpoint which was asked for, they chose only to guess a fraction which was close to a fraction already placed. This method of closer and closer approximations sometimes required three or four guesses; whereas taking an average would have hit on the first guess.

C who had been successful using the strips to help her determine moves on the numberlines mat game tried the strips here, but discovered that the

lines on the screen were much smaller. For a while after that she had no strategy but we willingly waited nearly five minutes for her to make a guess. When she finally made a guess and discovered you could pop a balloon with a "close" answer, and you did not need an "exact" answer, she relaxed and guessed more comfortably. With her good spatial perception, she was the only student to use the whole picture to figure out a middle point, but even she did not use this strategy consistently.

Seventh graders played Number Quest only briefly and then asked to play Darts again instead because they liked watching the dart travel across the screen. Darts raised some questions about relative size and equivalence. S, who knew a great deal about fractions but tended to jump to conclusions or to ask for help without taking time to think, complained, "This computer is broken. I put in two and two twooths, two and three threeths, two and four fourths and it keeps staying the same." Thinking that the computer or program was broken was a reasonable conclusion because up to that time we had tried only programs which had major bugs in them. I suspect that once S is convinced that the computer is not broken, he would enjoy hours of playing these sorts of computer games to learn about fractions.

In general, the students played Number Quest less cooperatively than Darts. Frequently one player made all the guesses in a particular round, but some fifth graders did enjoy taking turns for a round while the others observed. This gave F a chance to shine as a teacher. She sat back and asked leading questions and gave a few hints to E and to B. She showed enormous patience as well as skill at knowing just how much of a hint to give at critical moments.

B was intrigued with this game. She used the numberlines mat to help decide guesses. Her conservative strategies were successful here as they had been in the Island Hopping Computer Game. She repeatedly used a strategy of trying 1/2, 1/3, 1/4, ... or 1/2, 2/3, 3/4, ... These lists were quite apparent to her from the numberlines mat.

B seemed confident about her strategies, but I wondered if she would ever try to develop a strategy to find the number in fewer guesses. Perhaps she would assume that she was playing the best way and would eventually become bored. A few rounds of whole number Number Quest might make the strategy of picking the middle number clearer.

Summary notes

Except for E in fifth grade who cut classes and was reluctant to participate from the beginning, all of the students seemed to be engaged throughout these sessions. P in seventh grade missed classes because of illness. As the strongest math student in the group, he was able to join in whenever he returned. B in fifth grade missed half the classes for a number of reasons having to do with a loss of permission slips and family inconveniences. I was sorry about this because I felt she would have benefited enormously from the attention available in the small group, attention from F as well as from me.

The high points of the sessions were the Numberlines Mat Games, Darts, and Number Quest. All the students would have liked more time with these. I suspect that the Island Hopping Computer Games will also be popular when the bugs in the software are removed. The other activities provided important readiness for the games, in particular the initial estimating, the strip folding, the mixed number measurement, the stage three worksheets, and the writing of fractions in order. The discussions which resulted from students doing these activities together brought out many questions and answers about fractions. More time to play the computer games than we were able to provide would lead to additional shared experiences and further valuable discussions. After the initial introduction during the fractions unit, these games could be made available for students to play from time to time throughout the year.

Appendix A: Anecdotal Notes of Pilot Study - Part 2
Watertown, Massachusetts Public Schools
Anne Dickenson, research assistant

Stage 1: Simple Length Measurement: Integer Number of Units
Worksheets 1.1 and 1.2

The seventh graders began the session with nervous giggles and wondering glances to each other as I explained the kinds of things we were going to be doing and some of the reasons why. There were four girls and one boy; the girls seemed to break into two pairs of good friends, and the boy was a subdued singleton. The girls were conscious of each other and each other's work, and one girl in particular seemed very attuned to her friend and her friend's papers. Parts of the activities in Stage 1 seemed inappropriate to these seventh graders; there seemed to be a lack of interest and meaningful involvement at times. The activity of measuring a person's height seemed too young and inappropriate for the students. The girl who was being measured complained as the other students gave their estimates, "They're being mean to me." Most students used the pencil length to help figure out the estimates by drawing an imaginary line from the end of the pencil, comparing it to the straw, Cuisenaire rod, and paper strip. Their estimating skills seemed less developed than I had anticipated. I was surprised at the range of answers; there was little difference between the fifth and seventh graders. (For example, when the length of the windowsill in pencils was between 5 and 6, the estimates in the seventh grade were 6, 10, 5, 9, and 6 pencils.) This group knew and could articulate the relationship that the longer the unit, the less number needed to measure.

The fifth grade group had two boys and three girls in it. The students were fairly quiet and very cooperative during this first session. They worked in pairs to complete the measurement activities, and the session seemed to flow more smoothly than with the seventh graders. I am not sure if the ease of flow was due to the pairing of students or to the activities being more age appropriate. These fifth graders were fairly good at estimating. (For example, when the length of the table in pencils was between 9 and 10, the estimates were 10, 15, 14, 10, 12.) The pairing of students seemed to encourage conversations about the estimates. The relationship of the longer the unit, the fewer needed to measure was not immediately obvious to the fifth graders and took more prompting than in the seventh grade.

Stage 2: Measuring Strip Activities

Both groups had a difficult time with the folding of thirds and sixths. We did the folding together as a group, and I ended up re-doing at least half of their thirds rulers. The seventh graders had a better sense of the relationships between thirds and sixths and between halves, fourths, and eighths. "Fold a third in half and you get a sixth," remarked one seventh grader. The fifth graders did not initiate the same types of comments. The labeling process seemed trivial to just a few of the seventh graders. After laying down the halves and fourths parts one at a time, they anticipated the

directions and labeled their eighths, thirds, and sixths rulers at the fold lines without laying down the parts. For these few, I think that it was alright for them to omit that step, but for the other seventh graders and the fifth graders, the process of laying down, counting, and writing the fraction was important. Everyone was comfortable with the fraction notation - numerator/denominator. The black line across the top of the rulers was not as salient to the students as we had thought; some students began labeling along the bottom of the strip rather than along the line. The notion that they were labeling the end point of a length perhaps could have been made more explicit to the students. The seventh graders seemed to make a connection between the size of units from Stage 1 and the size of the fraction (e.g. the shorter the unit, the greater number of units needed to measure and so, the smaller the fraction, the more needed to make a whole). The fifth graders did not pick up on this as quickly. I am not sure if it indicates a lack of understanding about fractions or a lack of ability to make generalizations and to transfer learning to new situations.

Stage 2: Worksheets 2.1 - 2.4

Most students did not use their strips to label the points on Worksheet 2.1; some physically counted the marks and labeled, and others eyeballed the lines. Two of the fifth graders labeled 1 as 2/2 = 1, 3/3 = 1, etc. Worksheet 2.2 brought out more use of the rulers. One seventh grader noted the equivalency of points B and H. There were many conversations as to which ruler worked, which ruler "lined up." Worksheet 2.3 was perceived as "a little bit harder" than the others. All the students used their rulers for at least a couple of the lines on this sheet. One of the fifth graders had finished before the others, and I asked him to check and see if there were any points which had other names. He found that 3/4 and 6/8 were the same length.

The fifth graders seemed less interested in each other's work than the seventh graders did. The one seventh grade girl mentioned earlier was more or less copying the fractions from her friend's paper. I began to wonder how confident she felt with fractions and questioned her individually. She seemed nervous in answering.

We had some extra time in the fifth grade session and so we went over equivalency with worksheet 2.1. I would ask questions like: "Name a fraction equivalent to.. (e.g 3/4)"; "Name the fractions between... (e.g. 2/3 and 7/8)"; and "Name two fractions where there are no fractions in between on the sheet." Then the students took turns asking the same kinds of questions. The group seemed to enjoy this type of activity and began to understand some of the ordering properties.

One day after playing the Island Hopping Game, we came back to Worksheet 2.1 and looked at equivalency and order with additional tasks. We first went through the chart and named fractions with equivalent fractions; many remembered them from the last time. I then named two fractions and asked which was bigger. The first couple of examples were difficult for the students. Then the students named the two fractions for other students to answer. One girl named two equivalent fractions and said, "I tricked you." That led into another round where one student named a fraction that had an equivalent and another had to name the equivalent. The next activity sparked

an interesting discussion. I asked them to name a fraction where there was no equivalent fraction on the sheet. I commented that they named eighths a lot and one replied, "There are more of them." A discussion ensued about what fraction is smaller than 1/8. They went from tenths, twelfths, twentieths, fortieths, to millionths.

After two or three sessions both groups were fairly comfortable with me and the format of the sessions. The seventh graders became less cool, and the fifth graders became more animated. The tape recorder was a novelty at first but the students soon forgot it was there. The students came eagerly to the sessions, and I think liked the variety of activities.

Stage 2: Island Hopping Games

I began by demonstrating how to play the game by modeling a few turns. The students used the oaktag rulers. Both groups played two versions (1 and 2) of the game during the first session and expressed their preference for the second version. The conversations and extensions of playing rounds of the game were important as they engaged the students in thoughts about relative size of fractions and about equivalency. Snatches of conversations included "2/2, you have to use your whole halves ruler"; "Oh good, I made it to an island with 3/8"; "2/8 is the same as 1/4"; and "Hey, I can use 3/4 or 6/8." Some of the thinking emerged from the natural flow of the game; some was provoked by my questions; and some grew out of questions and comments of other students.

During the second session with the Island Hopping Game, after a general introduction to the computer version, the group split into two groups, three students playing with the paper maps and two playing the computer game. Then they switched. Having two activities happening at once worked well for the students, but it was difficult for me to observe both games. Students played the pass-along version of the paper game (Version 3) and the planning version (Version 5). The pass-along version seemed to be more fertile for thoughts about fractions. Particularily for the fifth graders, focusing on choosing one fraction at a time seemed more meaningful than attempting to create a realistic plan with five different fractional lengths. The seventh graders were better planners in general.

At a later session the fifth graders played another round of the Island Hopping Game. This was after they had used fifths and tenths rulers. We came up with a new version which used the idea of a "wild card." If you got one of the 0/n fractions, then you could choose your own fraction with denominator 5 or 10 and use a fifths or tenths ruler. This version added some spice.

Stage 2: Island Hopping Computer Games

The computer game presented some different challenges to the students, though it was slower moving than the paper versions. Inputting their own fraction was the most favorite version of the game; there the students could use fractions with denominators through 10 rather than the limited denominators (2, 3, 4, 6, 8) in the paper game. When the seventh grade group was asked to react to the computer version, comments included "It was slower,

maybe it should have a timing element"; "I liked typing"; "Have a sea monster come and get you"; and "I liked the paper game better." The software in its prototype form did have some flaws that were potentially frustrating to the students. Moving the cursor around the circle was awkward; indicating the other player's turn was somewhat confusing; and the ending of the game was ambiguous. However, some of the bugs in the program tended to provoke interesting questions. For example, when some of the seventh graderas played the version with the different units, at times the program failed to switch units and students were able to observe and complain about the discrepancies. Using two units was difficult for the fifth graders; that option did not seem to make sense to them.

When I demonstrated to the fifth graders the version of the Island Hopping Game where the players type in their own fractions, the first player typed in 1/2. The second person typed in 2/2. Those moves were recorded on the screen. The first player then saw and declared that she needed a fraction that was bigger than 1/2 but less than 2/2 to reach an island. It took a bit of prompting by me to get them to remember which fractions might be between 1/2 and 2/2. All of those wonderful activities we had done with Worksheet 2.1 seemed easily forgotten. The same girl sighed, "I know them, I just can't think of what they look like, how long they are."

Again, as with the paper game, I think the computer version fostered thoughtful comments about the relative size of the fractions used. Remarks such as the following were heard: "That's not a halves circle, that's a sixths circle"; "a seventh is smaller than an eighth"; "an eighth is bigger...no it isn't"; "a fifth, that's pretty small..well so isn't a ninth"; "I got 1/2 and 1/6. How big is 1/6?"; and "1/3 is bigger."

Stage 3: Part - Whole Relationships Activities
Worksheets 3.1 and 3.2

I explained the difference between starting with a whole and finding the parts and starting with a part and finding the whole. In each group one student pointed out that they had gone from parts to whole when they laid down the fraction parts to label the paper strips. I was pleased to see the students make that connection. Before handing out Worksheets 3.1 and 3.2, I modeled a few tasks by posing questions like "If this is 1/4, would you know how to find the whole?" while holding up a red rod. Initial answers were "Yeah, three times as long...no four times." The students found the answers to questions I asked and then posed questions for each other to solve. A notion I stressed prior to working on the Worksheets was that the same physical length, like a red rod, can be a third or a half when the whole is different. You need to ask "a third or a half of what?"

Worksheet 3.1 was easier than 3.2. The students didn't have trouble either drawing the whole or fractional part (e.g. 4/5). One student remarked about the second half of the sheet, "That's easy; draw just the part; you do it the same way." At first they had to be reminded to draw on the line below (and not just extend the line with the given fraction) and to label the end points. The whole end points had varied notations: 1 (4 students), 3/3 (5 students), and 1/whole (1 student). About half the students used iteration and the other half laid down the neccesary number of rods. I was surprised

that everyone started measuring at the 0 and didn't continue from the given fraction. It reminded me of "beginning adders" who, when given a problem of 5 + 3, start counting at 1, count up to 5, and then add 3 more. More "experienced adders" begin adding from 5. Having two separate lines might have actually encouraged the starts at 0. Many students asked, "How could 1/7 be longer than 1/6 and 1/4?" The physical layout of the sheet was confusing. I reminded them that the whole unit was different, but my words were not compelling enough. These worksheets will need careful review.

Again, I modeled the problems on Worksheet 3.2 before handing out the sheet. Posing the question, "If this is 2/3, how long is the whole?" The response was "You need one more third." So the next problem was how to find 1/3. We did a few more without the sheet and then did the first one together in the seventh grade group. Three of the seventh graders whipped through the worksheet, but two of them needed help thinking about the problems, particularly the last one. Since there was some bewilderment on the fifth graders' faces, we did the first two together. The initial response of the fifth graders when working on the second problem (If this is 5/8, draw one whole) was that you needed five more. My sense was that the fifth graders had a tough time with this sheet. The last problem was indeed a challenge for all but the pro in the group that day. One fifth grader solved it with no prompting, but the rest needed questions like, "Can you think of another way to say 2/3?"

Stage 3: Numberlines Mat Activities and Games

We began with some activities using the mats - placing folded rulers on the appropriate lines, finding equivalent fractions, placing fingers down and naming the fractions between, and matching improper fraction cards with the mixed number points. There seemed to be a loss of understanding from similar activities with Worksheet 2.1. The responses to questions like which is bigger, 1/2 or 1/3 were not immediately found. I am not sure if the previous sheets had confused the issue or if the transfer from Worksheet 2.1 needed to be explained much more explicitly.

Taking Cornelia Tierney's suggestion of using pennies as markers, we began playing some Numberlines games. I had never seen the seventh grade group quite as involved as they were playing these games. We had two games going on at once and hence the tape recording of this session was difficult to understand as there was so much animated conversation. The one girl I had been worried about really did well while playing these games. Another girl made a marvelous suggestion for a new game where the fractions could be split up on a move. For example, if the card drawn was 8/6, a player could move 1/2 on the halves line, 2/3 on the thirds line, and 1/6 on the sixths line. I was amazed how quickly two of the seventh graders picked up on this splitting. The other two girls (the boy was absent) chose not to split up the fractions the first time, but a few days later we played again and everyone got into the act, even the boy who had been absent. The fifth grade group was not as enthusiastic as the seventh graders though they certainly enjoyed the games. In the fifth grade, there were no innovations suggested, but we did play the up-dated versions suggested by the seventh graders.

Stage 3: Numberlines Computer Games

The second time playing, we also had a Numberlines game on the computer. In its early stage of development, this piece of software was shakey and at times the program aborted without any warning. In comparison with the mats, the computer version was much slower; it also didn't have pennies to win. The purpose of putting it on the computer was to provide different kinds of lines, rather than just the seven on the mat. The students used the Numberlines Mats to help while playing. Based on this piloting, though limited, I think that the mats are much more engaging than the software, and more thought needs to be given to a possible computer numberlines program.

Stage 3: Mixed Number Measurement Activities
Worksheet 3.3

The students only completed Worksheet 3.3 because I had not made the correct cardboard lengths for Worksheet 3.4. This activity was somewhat frustrating because of the students' inaccuracies in measuring. In both groups there were students who did not get to appreciate the equivalent fractions because of sloppy measurement. I think the idea of the sheets is good and those who indeed found that a length measures both 1 3/4 and 1 6/8 were impressed. Perhaps the measuring tapes made in Stage 1 could be used. One or two seventh graders predicted the equivalencies before measuring. The wording of "Does the ruler work?" seemed confusing to the fifth graders. Both groups, when asked if they only had one ruler to use, answered they would choose the tenths ruler because it was more exact.

Stage 3: Mixed Number Measurement Software

This game was demonstrated to the whole group, and because of time, played in a group with students taking turns at the keyboard. The strategies were not very well articulated. One fifth grade girl always began with tenths because "there are more little divisions...you have a better chance to get it." Other students used the Numberlines Mat to figure out the measurement. I tended to interject clarifications like "It's between 1 3/4 and 1 7/8. Can you think of a fraction between those?" as the students were beginning to play. There is still a serious bug in this program; it will not accept equivalent fractions when determining the type of measuring strip. This caused confusion with the students.

During a second session when a small group of fifth graders were playing the measurement game, one boy spontaneously ran and got the fraction strips to help the group figure out the lengths. The girls at the computer would say "It's between 2/3 and 3/3" or even "It's between 2/3 and 3/4". The boy would line up the rulers and pick appropriate ones to suggest a next try. This strategy began to work a few times, but then the students ran into the non-equivalent bug in the program and the girls unfortunately became discouraged and non-believers that the boy's use of rulers was helpful.

Stage 4: Order Relationships of Fractions and the Notion of Betweenness
Darts (Commercial Software)
Number Quest (Commercial Software)

The commercial software, Number Quest and Darts, certainly was more polished and less aggravating to use than the prototype software. What was wonderful to see was the use of the numberlines mats while playing both games. I demonstrated each game first and then split the group up to play on their own. The seventh graders felt that Number Quest was too easy. While demonstrating Number Quest to the fifth graders, I asked what fraction they would guess first. Answers were "1/4, 1/6, and then 1/8." I then asked what would they respond if they wanted to find a number between 0 and 100. The answer was to "Start with 50," and the answer to "Why 50?" was because "It was half way." "O.K., what would be half way between 0 and 1?" was the next question, and the students said to "Start with 1/2." That binary search strategy, enhanced by the numberlines mats, was then employed by all the students while playing Number Quest.

The students found that Darts had a lot of action during a game; watching the arrow move across the screen was motivating. The students were not as successful with Darts as they were with Number Quest, but they liked playing it better. Both groups expressed a preference for Darts because there was more to do in a turn. The use of mixed numbers, three balloons to pop, and changing unit size after each round creates variety. Students used the Numberlines Mats well with this game too.

Summary Notes

There are a few trends which cropped up throughout the sessions which merit noting. I could see some growth in particular students' understandings; of course that is what I wanted to see. The talking about fractions seemed easier and more confidently expressed by all students. The types of activities done here fostered students' learning from each other which is a good thing. Because of the variety of tasks, different students could shine at different times. There was one especially sharp seventh grader, but otherwise there was a nice shifting of competence and interest level as the activities varied. The activities within a concrete context (like the games) were well received; students were more involved in these kinds of activities than in a more abstract context (like the parts - wholes activities). Better conversations about fractions happened in the concrete settings.

A seventh grader said that she liked the strips and Numberlines Mat because they helped her "represent and visualize the fractions." I think that her articulate comment was felt by all the students. The one fifth grader who said, "I know them, I just can't think of what they look like, how long they are" was expressing the same notion. Somehow we need to provide students with better visual models of fractions. A large wall chart similar to the Numberlines Mat would be a good start.

When asked how did they had liked the sessions, one fifth grader answered, "It's different..we've had only boring fraction games before."

APPENDIX B

Guidelines for Observers
ETC Fraction Group Pilot Study

Name of Observer _____ Date _____

Group Observed _____ School _____

Please respond to the following types of issues as you observe. Record your notes in whatever format you feel is most comfortable for you and least distracting to the students.

a) What aspects of the lesson were too obvious to the students or went too slowly?

b) Were there any gaps in the development of the lesson and if so, what changes would you recommend?

c) What did you feel the students learned?

d) What were the high/low points of the lesson?

e) What made the lesson work or not work?

f) Discuss the motivation and interest level of each student.

g) List pertinent quotes of the students.

h) Comment on any other points that you feel would be helpful to this pilot study.

It would also be appreciated if you could stay for 15 - 20 minutes immediately following the observation to talk about the lesson with the person doing the piloting. List any further thoughts that you feel would be useful and then leave your comments with the person doing the piloting.

Thank you,

The Fractions Group

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